Art. I.—*The Alternate Current Transformer.*

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(With Plates I.-V.).

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The following paper is divided into three Sections.

In Section I. the mathematical theory of the closed-circuit transformer for sinusoidal wave forms is developed, and reduced to a form suitable for practical application.

In Section II. is given an example of the application of the practical formulae obtained in Section I. to the design of transformers to operate different classes of load.

Section III. contains analytical investigations relating to magnetic leakage in transformers, to what are called the transformer numerics, and to the determination of the most efficient shapes of transformers of different types as well as a general solution of the transformer problem in which no assumptions with regard to leakage are made.

Section I.

1. It is well known that when an alternate magneto-motive force (M.M.F.) operates in a magnetic circuit (laminated), the M.M.F. per unit length (H, say) and the average flux density (B, say) can be expressed as follows :—

$$H = H_1[\text{Sin} wt + h_3 \text{Sin} 3(wt - y_3) + \&c.]$$

$$B = B_1[\text{Sin}(wt - \delta) + b_3 \text{Sin} 3(wt - \beta_3) + \&c.]$$

where the period is $2\pi/w$; and that the iron losses per cycle, per unit volume, due to hysteresis and eddy currents are equal in this case to

$$\frac{H_1 B_1}{4}[\text{Sin}\delta + 3h_3 b_3 \text{Sin} 3(\beta_3 - \gamma_3) + \&c.]$$

If $B_1 = \mu_0 H_1$ then μ_0 and δ will depend on B_1, w, and the wave form of H, as well as on the quality of the iron and the thickness of the laminae.

In some experiments on good transformer iron of thickness .04 cm. (q.p.), I have found by means of my wave tracer[1] that μ_0 and δ are given in terms of B_1 for periods .03 and .06 sec. by the curves shown in Fig. I., where the curves giving the corresponding iron losses are also shown.

In these experiments the wave forms of H were peaked, that is, the value of H at the crest was greater than the amplitude H_1 of its first harmonic. When the wave form of H is flat-topped, both μ_0 and δ are smaller for the same values of B_1 and w.

The values of B at the crests of the flux waves, corresponding to different values of B_1 the first harmonic, for the period .03 sec. are also given in Fig. 1, by the upper row of figures along the axis of x.

When the third and higher harmonics of H and B are neglected, the above equations take the simple forms,

$$\frac{\text{M.M.F.}}{\text{Length}} = \text{H}\,\text{Sin}\,wt$$

$$\frac{\text{Flux}}{\text{Section}} = \text{B}\,\text{Sin}(wt - \delta)$$

$$\text{B} = \mu\text{H}$$

$$\frac{\text{Iron losses per cycle}}{\text{Volume of iron}} = \frac{\text{HB}}{4}\text{Sin}\,\delta$$

$$\frac{\text{Iron losses per second}}{\text{Volume of iron}} = \frac{w\text{B}^2}{8\pi\mu}\text{Sin}\,\delta$$

which relations will be used in the following approximate theory of the transformer.

Notation.—The different periodic quantities considered will in the text be represented by letters such as $\overline{E_1}$, $\overline{C_1}$, $\overline{E_2}$, $\overline{C_2}$, with bars over them when the conception of both their amplitudes and phases is involved, while the amplitudes of these quantities will be represented by the same letters without the bar. Letters with the number 1 subscribed will refer to the primary, and with the number 2 subscribed to the secondary circuit.

The period of the alternations will be $2\pi/w$.

2. On the vector diagram, Fig. 2, let OR represent in amplitude and phase the resultant flux $\overline{F}$ looped on both the primary and secondary coils of a transformer.

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1 Phil. Mag., Nov., 1903.

This flux is produced by the ampere-turns

$\overline{n_1 C_1 + n_2 C_2}$ (a vector sum)

so that Amp. $\overline{F}$ or $F = \sigma \text{amp.}(\overline{n_1 C_1 + n_2 C_2})$

where $\sigma = 4\pi \times$ permeance of the magnetic circuit.

$$= 4\pi\mu \frac{\text{Section of iron}}{\text{Mean length of iron}}, \text{ for a closed circuit,}$$

and $\overline{F}$ is behind $\overline{n_1 C_1 + n_2 C_2}$ in phase by an angle δ. (See § 1).

[It will be shown that, throughout the range of operation of a transformer, when the primary volts and frequency are fixed, F is very nearly constant, so that δ and μ will be very nearly constant. On referring to Fig. 1, it will be seen that δ is fairly constant in any case in the neighbourhood of the flux densities generally used in transformers, and though, at the same densities μ is changing rapidly, we shall not, on account of the approximate constancy of F, introduce much error by assuming both δ and μ constant during the operation of the transformer.]

Hence from O draw OM, ahead of OR by the angle $\delta = \text{ROM}$, and in length equal to F/σ.

OM fully represents $\overline{n_1 C_1 + n_2 C_2}$.

In addition to the magnetic lines forming the main flux $\overline{F}$ and looped on both circuits of a transformer, there are others, the leakage lines, which are only partially looped on the circuits and whose action must be taken account of.

In Section III. of this paper it will be shown that, after the transformer is somewhat loaded, the effect of these leakage lines on its operation is the same as would be produced by two fluxes; one, the primary leakage flux, in phase with the primary current, and supposed to consist of lines that are looped on the primary circuit and miss the secondary, and the other, the secondary leakage flux, in phase with the secondary current and supposed to consist of lines that are looped on the secondary and that miss the primary circuit.

Let these two fluxes be specified by $x_1 \sigma n_1 \overline{C_1}$ and $x_2 \sigma n_2 \overline{C_2}$ where x_1 and x_2 are what we will call the *leakage coefficients* of the two circuits, and $\sigma = 4\pi \times$ permeance of the magnetic circuit as before.

3. The e.m.f. in the secondary coil being equal to

$$-n_2 \frac{d\overline{F}}{dt}$$

is represented by the vector RS equal in length to wn_2F and behind $\overline{F}$ in phase by a right angle.

For the present we will assume that the secondary current $\overline{C_2}$ lags behind the internal secondary e.m.f. $\overline{E'_2}$ or RS by an angle ϕ'. This angle will depend on the load and its power factor, as well as on the secondary magnetic leakage, and will subsequently be expressed as a function of these quantities.

From R draw RP making the angle $SRP = \phi'$ and drop SP perpendicular to RP; then the vector RP fully represents R_2C_2, where R_2 is the total resistance or its equivalent in the secondary circuit.

4. From M draw MN parallel to RP and equal to n_2C_2, that is

$$MN = n_2C_2 = n_2\frac{RP}{R_2} = n_2\frac{wn_2F\text{Cos}\phi'}{R_2} = \theta'\text{Cos}\,\phi'\frac{F}{\sigma}$$

where $\theta' = \dfrac{wn_2^2\sigma}{R_2}$,

then the vector NM represents $\overline{n_2C_2}$, and as OM represents $\overline{n_1C_1 + n_2C_2}$, we have $\overline{n_1C_1}$ fully represented by the vector ON.

As the angle $OMN = \frac{\pi}{2} + \delta + \phi'$ and $OM = F/\sigma$ we find that

$$ON \text{ or } n_1C_1 = \Delta'\frac{F}{\sigma}, \quad = \frac{\Delta'}{\theta'\text{Cos}\phi'}n_2C_2$$

where $\Delta'^2 = 1 + 2\theta'\text{Cos}\phi'\text{Sin}(\delta + \phi') + \theta'^2\text{Cos}^2\phi'$.

If the angle MON be called χ, we find, by projecting the sides of the triangle OMN on OR and on a line perpendicular to OR, the relations

$$\Delta'\text{Cos}(\chi + \delta) = \text{Cos}\delta + \theta'\text{Cos}\phi'\text{Sin}\phi'$$
$$\Delta'\text{Sin}(\chi + \delta) = \text{Sin}\delta + \theta'\text{Cos}^2\phi'$$

which will be useful.

5. From O along ON cut off a length OB that will represent $\overline{r_1C_1}$, where r_1 is the resistance of the primary coil; OB will represent therefore the effective e.m.f. that produces current in the primary coil, and will be the vector sum of (*a*) the impressed e.m.f. $\overline{E_1}$, (*b*) the e.m.f.

$$-n_1\frac{\overline{dF}}{dt}$$

due to variation of the main flux $\overline{F}$, and (*c*) the e.m.f.

$$-n_1\frac{d}{dt}(x_1\sigma n_1\overline{C_1})$$

due to variation of the primary leakage flux.

Hence from B draw BC perpendicular to ON and equal to

$wx_1n_1^2\sigma C_1 = x_1\tau_1r_1C_1$ say,

where $\tau_1 = \dfrac{wn_1^2\sigma}{r_1}$

CB will fully represent (c).

From C draw CE perpendicular to OR and equal to wn_1F, EC will fully represent (b).

Join OE. OE will fully represent E_1, the e.m.f. impressed on the primary of the transformer.

6. At this place attention may be drawn to the importance that will be attached in what follows to the quantities τ_1, τ_2, and θ'.

As τ_1 which we will call the *numeric of the primary circuit of the transformer* or, shortly, the *primary numeric*, is equal to

$$w\frac{n_1^2\sigma}{r_1}$$

and $\dfrac{n_1^2}{r_1} = \dfrac{n_1^2}{\rho n_1\frac{l_1}{a_1}} = \dfrac{n_1a_1}{\rho l_1}$

where a_1 = sectional area of primary wire

l_1 = mean length of primary turns

ρ = specific resistance of copper

also $\sigma = 4\pi$. permeance of magnetic circuit. We see that τ_1 *is equal to* $4\pi w$ *into the conductance of the primary wires, considered as one turn or belt, into the permeance of the magnetic circuit.*

τ_2 is a similar constant for the secondary circuit, and will generally be nearly equal to τ_1; we will call it the *secondary numeric.* In what follows the ratio of τ_1 to τ_2 will where necessary be denoted by f so that

$$f = \frac{\tau_1}{\tau_2} = \frac{\frac{n_1^2}{r_1}}{\frac{n_2^2}{r_2}} = \frac{\frac{n_1a_1}{l_1}}{\frac{n_2a_2}{l_2}}$$

On the other hand θ' is a variable, varying with the load on the transformer, and for a given load-power-factor approximately as the load.

R_2 being the total resistance or its equivalent in the secondary circuit

$$\theta' = \frac{wn_2{}^2\sigma}{R_2} = \frac{r_2\tau_2}{R_2}$$

It is worth noting that τ_1, τ_2, and θ' are of zero dimensions.

7. Returning to the diagram Fig. 2, if we call the angle EOB α, so that Cosα is the power factor of the transformer, we find by projecting the figure OECB on ON and on a line perpendicular to ON, that

$$E_1\text{Cos}\alpha = r_1C_1 + wn_1F\text{Sin}(\delta+\chi) = r_1C_1[1 + \frac{\tau_1}{\Delta'}\text{Sin}(\delta+\chi)]$$

$$= r_1C_1[1 + \frac{\tau_1}{\Delta'^2}(\text{Sin}\delta + \theta'\text{Cos}^2\phi')]$$

making use of the relations in §§ 4 and 5.

$$E_1\text{Sin}\alpha = x_1\tau_1r_1C_1 + wn_1F\text{Cos}(\delta+\chi) = r_1C_1[x_1\tau_1 + \frac{\tau_1}{\Delta'}\text{Cos}(\delta+\chi)]$$

$$= r_1C_1[x_1\tau_1 + \frac{\tau_1}{\Delta'^2}[\text{Cos}\delta + \theta'\text{Cos}\phi'\text{Sin}\phi')]$$

whence, squaring and adding

$$E_1{}^2 = r_1{}^2C_1{}^2\left\{1 + x_1{}^2\tau_1{}^2 + \frac{\tau_1{}^2}{\Delta'^2} + \frac{2\tau_1}{\Delta'}[\text{Sin}(\delta+\chi) + x_1\tau_1\text{Cos}(\delta+\chi)]\right\}$$

or $C_1 = \frac{E_1}{r_1\tau_1}\frac{\Delta'}{D'}$

where $D'^2 = 1 + 2x_1\text{Cos}\delta + 2\frac{\text{Sin}\delta}{\tau_1} + 2\theta'\text{Cos}\phi'(x_1\text{Sin}\phi' + \frac{\text{Cos}\phi'}{\tau_1}) + \Delta'^2\left(x_1{}^2 + \frac{1}{\tau_1{}^2}\right)$

Dividing E_1Sinα by E_1Cosα

$$\tan\alpha = \frac{\text{Cos}\delta + \theta'\text{Cos}\phi'\text{Sin}\phi' + x_1\Delta'^2}{\text{Sin}\delta + \theta'\text{Cos}^2\phi' + \frac{1}{\tau_1}\Delta'^2}$$

The above relations enable us to determine practically τ_1 and δ for any closed-circuit transformer. For on open secondary $\theta' = 0$.

$$\Delta' = 1. \quad D' = 1 + x_1\text{Cos}\delta + \frac{\text{Sin}\delta}{\tau_1} = 1 \text{ (q.p.)},$$

as it will be shown later on that x_1 is always a small fraction and τ_1 a large number for a transformer of the type treated in this section. Hence if C_0 be the primary current on open secondary

$$\tau_1 = \frac{E_1}{r_1 C_0\left(1 + x_1 \operatorname{Cos}\delta + \frac{\operatorname{Sin}\delta}{\tau_1}\right)} = \frac{E_1}{r_1 C_0}$$

[The same is obvious otherwise, for

$$r_1\tau_1 = w n_1^2 \sigma = w L_1$$

where L_1 is the inductance of the primary on open secondary].

Also on open secondary as $\theta' = 0$, &c.

$$\tan\alpha = \frac{\operatorname{Cos}\delta + x_1}{\operatorname{Sin}\delta + \frac{1}{\tau_1}} = \operatorname{Cot}\delta \quad \text{(q.p.)}$$

or $\operatorname{Cos}\alpha = \operatorname{Sin}\delta$

that is, *the power factor of a closed circuit transformer on open secondary is equal to the sine of the angle of magnetic retardation of its iron for the period aud flux density used.*

8. In the diagram Fig. 2, we see that $\overline{C}_2$ is behind $\overline{C}_1$ in phase by an angle $\pi - \beta$ where $\beta = ONM$.

Projecting ON on NM and on a line perpendicular to NM, we find that

$$\Delta' \operatorname{Cos}\beta = \operatorname{Sin}(\delta + \phi') + \theta' \operatorname{Cos}\phi'$$

$$\Delta' \operatorname{Sin}\beta = \operatorname{Cos}(\delta + \phi').$$

Also if $\overline{C}_2$ be behind E_1 in phase by an angle $\pi + \lambda$ we see that $\lambda = \alpha - \beta$.

9. The amplitudes of the different quantities can now be written down in terms of E_1 as follows :—

$$C_1 = \frac{E_1}{r_1\tau_1} \frac{\Delta'}{D'},$$

$$w n_1 F = \frac{E_1}{D_1}$$

$$\text{Amp } (\overline{n_1 C_1 + n_2 C_2}) = \frac{F}{\sigma} = \frac{n_1 C_1}{\Delta'} = \frac{n_2 C_2}{\theta' \operatorname{Cos}\phi'} = \frac{E_1}{r_1\tau_1} \frac{n_1}{D'}$$

and if E'_2 be the total e.m.f. generated in the secondary,

$$E'_2 = w n_2 F = \frac{n_2}{n_1} \frac{E_1}{D'},$$

The total power P'_2 developed in the secondary

$$= \tfrac{1}{2} E'_2 C_2 \operatorname{Cos}\phi' = \tfrac{1}{2} r_1 C_1^2 \frac{\theta' \tau_1 \operatorname{Cos}^2\phi'}{\Delta'^2} = \frac{E_1^2}{2 r_1 \tau_1} \frac{\theta' \operatorname{Cos}^2\phi'}{D'^2}.$$

10. If $\overline{E}_2$ be the terminal e.m.f. of the secondary and $\operatorname{Cos}\phi$ the power factor of the load, the relations connecting these

quantites with $\overline{E'_2}$, $\text{Cos}\phi'$, etc., can now be obtained as follows:—

From SP Fig. 2 cut off ST so that

$$ST = wx_2n_2^2\sigma C_2 = x_2\theta' R_2 C_2$$

then ST represents $-n_2\frac{d}{dt}(x_2\sigma n_2\overline{C_2})$

that is, the e.m.f. in the secondary due to variation of its leakage flux.

From RP cut off $RQ = r_2C_2$, then RQ represents the ohmic drop in the secondary.

Subtracting the vectors ST and RQ from RS (which represents the total e.m.f. in the secondary), we get QT, which fully represents $\overline{E_2}$, the terminal e.m.f., and the angle $PQT = \phi$ where $\text{Cos}\phi$ is the power factor of the load.

If R be the *external* resistance or its equivalent in the secondary circuit

so that $R = R_2 - r_2$, and if

$$\theta = \frac{wn_2^2\sigma}{R}$$

then as $\theta' = \frac{wn_2^2\sigma}{R_2}$ and $\tau_2 = \frac{wn_2^2\sigma}{r_2}$

we have $\frac{1}{\theta'} = \frac{1}{\theta} + \frac{1}{\tau_2}$. (I.)

Since $\frac{QP}{RP} = \frac{R_2 - r_2}{R_2} = \frac{R}{R_2} = \frac{\theta'}{\theta}$

$$E_2\theta\text{Cos}\phi = E_2'\theta'\text{Cos}\phi'$$

or $E_2 = \frac{\theta'\text{Cos}\phi'}{\theta\text{Cos}\phi}\,\frac{n_2}{n_1}\,\frac{E_1}{D'}$ (see § 9).

Again since $PS = PT + TS$

$$R_2C_2\tan\phi' = RC_2\tan\phi + wn_2^2\sigma x_2 C_2$$

$$\frac{\tan\phi'}{\theta'} = \frac{\tan\phi}{\theta} + x_2 \quad . \qquad \text{(II.)}$$

By means of the relations I. and II. we can now transform the formulæ already obtained in θ' and ϕ' to others in θ and ϕ.

11. Before doing so, however, it will be well to direct attention to the possible values of τ_1, τ_2, θ, θ', x_1, x_2, and $\text{Sin}\delta$, as when these are considered the formulæ admit of considerable simplification through dropping terms of negligible value.

The constants τ_1 and τ_2 for a transformer of 1 K.W. capacity at 50 periods would in no case be less than 1200, and it will be shown in Section III. (§ 55), that for similar transformers they are proportional to the square root of the output, and to the square root of the frequency.

The greatest practical value of θ or θ' for any transformer will not be much above

$\sqrt{\frac{\tau \text{Sin}\delta}{2}}$ where τ is the mean of τ_1 and τ_2

unless in case of excessive overload.

The leakage coefficients x_1 and x_2 should each be less than .002; and in transformers whose coils are wound in sections and interleaved they become very much smaller.

δ, the angle of magnetic retardation, will lie between 40° and 55°, its value depending on the quality of the iron, thickness of laminæ, flux density and frequency; hence Sinδ will have a value between .65 and .8.

[The formulæ given in this paper are only roughly approximate when applied to open-circuit transformers, as will be explained further on.

For them the τ constants are roughly .05 – .04 times the constants of closed-circuit transformers of the same capacity while Sinδ=.15—.08.]

12. Since $\frac{1}{\theta'^2 \text{Cos}^2\phi'} = \frac{1}{\theta'^2} + \frac{\tan^2\phi'}{\theta'^2}$

we find on substituting for $\frac{1}{\theta'}$ and $\frac{\tan\phi'}{\theta'}$ from equations I. and II. of § 10 that

$$\frac{\theta \text{Cos}\phi}{\theta' \text{Cos}\phi'} = \text{M}$$

where $$\text{M}^2 = 1 + 2\left(x_2 \text{Sin}\phi + \frac{\text{Cos}\phi}{\tau_2}\right) \theta \text{Cos}\phi + \left(x_2^2 + \frac{1}{\tau_2^2}\right) \theta^2 \text{Cos}^2\phi$$

From § 4 we have

$$\frac{\Delta'^2}{\theta'^2 \text{Cos}^2\phi'} = 1 + \frac{2\text{Sin}(\delta + \phi')}{\theta' \text{Cos}\phi'} + \frac{1}{\theta'^2 \text{Cos}^2\phi'}$$

$$= 1 + \frac{2\text{Sin}\delta}{\theta'} + 2\text{Cos}\delta \frac{\tan\phi'}{\theta'} + \frac{1}{\theta'^2} + \frac{\tan^2\phi'}{\theta'^2}$$

substituting as before we find that

$$\frac{\Delta'^2}{\theta'^2 \text{Cos}^2\phi'} = \frac{\Delta^2}{\theta^2 \text{Cos}^2\phi}$$

where $\Delta^2 = \theta^2\text{Cos}^2\phi\left(1 + 2x_2\text{Cos}\delta + 2\frac{\text{Sin}\delta}{\tau_2}\right) +$

$$\theta\text{Cos}\phi\text{Sin}(\delta + \phi) + \text{M}^2$$

in which, for all practical purposes, M^2 may be taken $= 1$.

Similarly $\frac{\text{D}'^2}{\theta'^2\text{Cos}^2\phi'} = \frac{\text{D}^2}{\theta^2\text{Cos}^2\phi}$

where, after dropping insignificant terms,

$$\text{D}^2 = 1 + 2x_1\text{Cos}\delta + 2\frac{\text{Sin}\delta}{\tau_1} + 2\theta\text{Cos}\phi(\text{X Sin}\phi + \text{T Cos}\phi) + \theta^2\text{Cos}^2\phi(\text{X}^2 + \text{T}^2)$$

in which $\text{X} = x_1 + x_2$

$$\text{T} = \frac{1}{\tau_1} + \frac{1}{\tau_2}.$$

We also find

$$\tan\alpha = \frac{\text{Cos}\delta + \theta\text{Cos}\phi\text{Sin}\phi + \text{X}\theta^2\text{Cos}^2\phi}{\text{Sin}\delta + \theta\text{Cos}^2\phi + \text{T}\theta^2\text{Cos}^2\phi}$$

$$\tan\beta = \frac{\text{Cos}(\theta + \phi)}{\text{Sin}(\delta + \phi) + \theta\text{Cos}\phi} \quad \text{(see § 8).}$$

$$\tan\lambda = \frac{\tan\phi + \theta\text{X}}{1 + \theta\text{T}}.$$

13. Transforming the equations in § 9 by means of the relations in § 12, we get

$$\text{C}_1 = \frac{\text{E}_1}{r_1\tau_1}\frac{\Delta}{\text{D}}$$

$$wn_1\text{F} = \frac{\text{M}}{\text{D}}\text{E}_1$$

$$\frac{1}{\text{M}}\frac{\text{F}}{\sigma} = \frac{n_1\text{C}_1}{\Delta} = \frac{n_2\text{C}_2}{\theta\text{Cos}\phi} = \frac{\text{E}_1}{r_1\tau_1}\frac{n_1}{\text{D}}$$

$$\text{E}_2 = \frac{n_2}{n_1}\frac{\text{E}_1}{\text{D}} \quad \text{(see § 10).}$$

$$\text{P}'_2 = \tfrac{1}{2}\frac{\text{E}_1^2}{r_1\tau_1}\frac{\theta\text{Cos}^2\phi}{\text{D}^2}\left(1 + \frac{\theta}{\tau_2}\right)$$

and if P_2 be the output of the transformer

$$\text{P}_2 = \tfrac{1}{2}\text{E}_2\text{C}_2\text{Cos}\phi = \tfrac{1}{2}\frac{\text{E}_1^2}{r_1\tau_1}\frac{\theta\text{Cos}^2\phi}{\text{D}^2}$$

also if H_1 and H_2 be the copper loss in the primary and secondary respectively,

$$\text{H}_1 = \tfrac{1}{2}r_1\text{C}_1^2 = \tfrac{1}{2}\frac{\text{E}_1^2}{r_1\tau_1^2}\frac{\Delta^2}{\text{D}^2}$$

$$H_2 = \tfrac{1}{2} r_2 C_2^2 = P_2' - P_2 = \tfrac{1}{2} \frac{E_1^2}{r_1 \tau_1 \tau_2} \frac{\theta^2 \text{Cos}^2 \phi}{D^2}$$

where M, $\triangle$, and D have the values given in the last paragraph.

14. When we multiply both sides of the first equation in § 7 by $\tfrac{1}{2} C_1$, we get

$$\tfrac{1}{2} E_1 C_1 \text{Cos}\alpha = \tfrac{1}{2} r_1 C_1^2 \left\{ 1 + \frac{\tau_1}{\triangle'^2} (\text{Sin}\,\delta + \theta' \text{Cos}^2 \phi') \right\}$$

which expresses the power P_1 supplied to the transformer as the sum of three terms, of which the first

$$\tfrac{1}{2} r_1 C_1^2 = H_1$$

is equal to the copper loss in the primary coil; the second term

$$\tfrac{1}{2} r_1 C_1^2 \frac{\tau_1 \text{Sin}\delta}{\triangle'^2} = H_3 \text{ say,}$$

is equal to the total iron loss in the transformer; and the third term

$$\tfrac{1}{2} r_1 C_1^2 \frac{\theta' \tau_1 \text{Cos}^2 \phi'}{\triangle'^2} = P_2' \text{ say.}$$

is equal to the power passed down to, and developed in the secondary coil:—

For, neglecting rC^2 losses, the energy entering the transformer on the primary side in any element of time dt is $\overline{n_1 C_1} \frac{d\overline{F}}{dt} dt$, and the energy leaving the transformer on the secondary side in the same element of time dt is $- \overline{n_2 C_2} \frac{d\overline{F}}{dt} dt$, hence in the time dt the transformer absorbs energy to the amount

$$\overline{n_1 C_1} + \overline{n_2 C_2} \frac{d\overline{F}}{dt} dt,$$

of which a part dM goes to increase the magnetic energy of the iron, while the remainder dW is dissipated as heat by hysteresis and eddy currents.

But amp. $\overline{n_1 C_1 + n_2 C_2} = F/\sigma$ so that we may write $\overline{n_1 C_1 + n_2 C_2} = \frac{F}{\sigma} \text{Sin} wt$ in which case $\overline{F} = F \text{Sin}(wt - \delta)$ where δ is the angle of magnetic lag:—

hence $dM + dW = w \frac{F^2}{\sigma} \text{Sin} wt \text{Cos}(wt - \delta) dt.$

Integrating over a complete period T, M returns to its original value and we get the core loss per cycle

$$W = \tfrac{1}{2} w T \frac{F^2}{\sigma} \operatorname{Sin}\delta$$

hence (as the core loss per second $H_3 = W/T$)

$$H_3 = \tfrac{1}{2} w \frac{F^2}{\sigma} \operatorname{Sin}\delta$$

$$= \tfrac{1}{2} r_1 C_1^{\,2} \frac{\tau_1 \operatorname{Sin}\delta}{\Delta'^2} \text{ (see § 9).}$$

In § 9, P'_2 the total power passed down to and developed in the secondary was shown to be equal to

$$\tfrac{1}{2} r_1 C_1^{\,2} \frac{\theta' \tau_1 \operatorname{Cos}^2\phi'}{\Delta'^2}$$

so that the different portions H_1, H_3 and P'_2 into which P_1 is divided are accounted for, and in § 13 are given the secondary copper loss H_2 and the output P_2 into which P'_2 is subsequently divided.

Transforming the above expression for H_3 to one in terms of θ and ϕ by § 13 we find

$$H_3 = \tfrac{1}{2} \frac{E_1^{\,2}}{r_1 \tau_1} \frac{M^2}{D^2} \operatorname{Sin}\delta$$

and collecting the other power expressions

$$H_1 = \tfrac{1}{2} \frac{E_1}{r_1 \tau_1^{\,2}} \frac{\Delta^2}{D^2}$$

$$H_2 = \tfrac{1}{2} \frac{E_1^{\,2}}{r_1 \tau_1 \tau_2} \frac{\theta^2 \operatorname{Cos}^2\phi}{D^2}$$

$$P'_2 = \tfrac{1}{2} \frac{E_1^{\,2}}{r_1 \tau_1} \frac{\theta \operatorname{Cos}^2\phi}{D^2}$$

$$P_1 = P_2 + H_1 + H_2 + H_3.$$

It is worth noting that, as $M/D = 1$ to the first order, the iron loss H_3, and the flux F will, to the same order, be constant throughout the range of operation of a transformer.

15. The efficiency η, of the transformer being

$$= \frac{P_2}{P_1} = \frac{P_2}{P_2 + H_1 + H_2 + H_3}$$

we have $$\eta = \frac{\theta \operatorname{Cos}^2\phi}{\theta \operatorname{Cos}^2\phi + \frac{\Delta^2}{\tau_1} + \frac{\theta^2 \operatorname{Cos}^2\phi}{\tau_2} + M^2 \operatorname{Sin}\delta}$$

$$= \frac{\theta \operatorname{Cos}^2\phi}{\Omega}, \text{ where } \Omega = \operatorname{Sin}\delta + \frac{1}{\tau_1} + \theta \operatorname{Cos}\phi \left\{ \operatorname{Cos}\phi + \right.$$

$$\frac{2\,\text{Sin}(\delta+\phi)}{\tau_1}+2\left(x_2\text{Sin}\,\phi+\frac{\text{Cos}\,\phi}{\tau_2}\right)\text{Sin}\delta\Big\}+\theta^2\text{Cos}^2\phi$$

$$\times\left\{\frac{1}{\tau_1}+\frac{1}{\tau_2}+\frac{2}{\tau_1}\left(x_2\text{Cos}\delta+\frac{\text{Sin}\delta}{\tau_2}\right)+\left(x_2^{\,2}+\frac{1}{\tau_2^{\,2}}\right)\text{Sin}\delta\right\}$$

To find the value of θ, for which η is a maximum when ϕ is constant, we note that η is of the form

$$\frac{\theta}{a+b\theta+c\theta^2}$$

which is a maximum when $\theta_2=a/c=\theta_0^{\,2}$ (say), and its maximum value is

$$\frac{1}{b+2a/\theta_0}.$$

Hence the value of θ for maximum efficiency is given by

$$\theta^2\text{Cos}^2\phi=\frac{\text{Sin}\delta+\dfrac{1}{\tau_1}}{\dfrac{1}{\tau_1}+\dfrac{1}{\tau_2}+\dfrac{2}{\tau_1}\left(x_2\text{Cos}\delta+\dfrac{\text{Sin}\delta}{\tau_2}\right)+\left(x_2^{\,2}+\dfrac{1}{\tau_2^{\,2}}\right)\text{Sin}\delta}$$

which for all practical purposes may be reduced to

$$\theta^2\text{Cos}^2\phi=\frac{\text{Sin}\delta}{\dfrac{1}{\tau_1}+\dfrac{1}{\tau_2}}$$

and the maximum efficiency is given to a sufficient approximation by

$$\eta(\text{max})=\frac{1}{1+\dfrac{2}{\text{Cos}\,\phi}\sqrt{\left\{\dfrac{1}{\tau_1}+\dfrac{1}{\tau_2}\right\}\text{Sin}\delta}+2x_2\tan\phi\,\text{Sin}\delta}$$

Note.—It is obvious that all the formulae we have obtained will apply to non-inductive loads when we make $\phi=0$, and to loads having capacity when we make ϕ negative.

16. From §15 we find that the ratio of the copper losses H_1+H_2 to the iron loss H_3 is

$$=\frac{\dfrac{\triangle^2}{\tau_1}+\dfrac{\theta^2\text{Cos}^2\phi}{\tau_2}}{\text{M}^2\text{Sin}\delta}$$

Putting in this expression for $\triangle$ and M their values given in § 12 and then substituting for θ its value at maximum efficiency, we find that this ratio is

$$=1-\mathrm{Sin}\,\phi\Big(x_2\mathrm{Sin}\delta-\frac{\mathrm{Cos}\delta}{\tau}\Big)\sqrt{\frac{2\tau}{\mathrm{Sin}\delta}}+\text{ terms of lower orders,}$$

where in the second term we take $\tau_1=\tau_2=\tau$.

Hence at maximum efficiency, when the load is non-inductive ($\phi=0$) the copper and the iron losses of a closed-circuit transformer are very approximately equal, and differ by a small amount given by the above formula when the load is inductive.

17. To determine the value of θ (θ_z say) for which the copper losses are z times the iron loss; we have (see § 14)

$$\frac{\triangle^2}{\tau_1}+\frac{\theta^2\mathrm{Cos}^2\phi}{\tau_2}=z\mathrm{M}^2\mathrm{Sin}\delta$$

from which, after substituting for $\triangle$ and M their values given in § 12, θ_z can in general be determined.

For practical purposes θ_z will be given to a high order of accuracy by

$$\theta_z\,\mathrm{Cos}\,\phi=\sqrt{\frac{z\,\mathrm{Sin}\,\delta}{\mathrm{T}}+\frac{z\,\mathrm{Sin}\,\delta}{\mathrm{T}}\Big(x_2\mathrm{Sin}\,\phi+\frac{\mathrm{Cos}\,\phi}{\tau_2}\Big)}-\frac{\mathrm{Sin}(\delta+\phi)}{2}$$

where $\mathrm{T}=\frac{1}{\tau_1}+\frac{1}{\tau_2}$

18. In § 13 it has been shown that the output

$$\mathrm{P}_2=\tfrac{1}{2}\frac{\mathrm{E}_1^2}{r_1\tau_1}\frac{\theta\mathrm{Cos}^2\phi}{1+2x_1\mathrm{Cos}\,\delta+\frac{2\mathrm{Sin}\delta}{\tau_1}+2\theta\mathrm{Cos}\phi\{\mathrm{X}\,\mathrm{Sin}\phi+\mathrm{T}\,\mathrm{Cos}\phi\}+\theta^2\mathrm{Cos}^2\phi(\mathrm{X}^2+\mathrm{T}^2)}$$

where $\mathrm{X}=x_1+x_2$ and $\mathrm{T}=\frac{1}{\tau_1}+\frac{1}{\tau_2}$

$$\text{Let } \mathrm{P}_0=\tfrac{1}{2}\frac{\mathrm{E}_1^2}{r_1\tau_1\Big\{1+2x_1\mathrm{Cos}\delta+2\frac{\mathrm{Sin}\delta}{\tau_1}\Big\}}$$

$$=\frac{\text{Iron loss on open secondary}}{\mathrm{Sin}\delta}\qquad\text{(see § 14)}$$

$$=\frac{\text{Power absorbed on open secondary}}{\text{Power factor of transformer on open secondary}}\quad\text{(q.p.)}$$

and take $y=\frac{\mathrm{P}_2}{\mathrm{P}_0\mathrm{Cos}\phi}$

so that y is proportional to the output;

also let

$$\mathrm{X}\mathrm{Sin}\phi+\mathrm{T}\mathrm{Cos}\phi=p$$

$$\mathrm{X}\mathrm{Cos}\phi-\mathrm{T}\mathrm{Sin}\phi=q$$

and the above equation in P_2 can be put into the form

$$y=\frac{\theta\text{Cos}\phi}{1+2p\theta\text{Cos}\phi+\theta^2\text{Cos}^2\phi(p^2+q^2)}$$

or

$$y=\theta\text{Cos}\phi[1-2p\theta\text{Cos}\phi+(3p^2-q^2)\theta^2\text{Cos}^2\phi]$$

Inverting this series we get

$$\theta\text{Cos}\phi=y[1+2py+(5p^2+q^2)y^2]$$
$$=y\text{D}_0^{\ 2}$$

$$[\text{where } \text{D}_0=1+py+\tfrac{1}{2}(4p^2+q^2)y^2]$$

a very important relation, as it will enable us to *transform all our formulae from the independent variable θ to what is the practically important independent variable, namely, the output of the transformer.*

19. Thus if we let

$$\text{C}_0=\frac{\text{E}_1}{r_1\tau_1\left(1+x_1\text{Cos}\delta+\frac{\text{Sin}\delta}{\tau_1}\right)}$$

= Primary current on open secondary,

the formulæ in § 13 become

$$\text{C}_1=\text{C}_0\sqrt{y^2\text{D}_0^{\ 2}\left(1+2x_2\text{Cos}\delta+2\frac{\text{Sin}\delta}{\tau_2}\right)+2y\text{Sin}(\delta+\phi)+\frac{1}{\text{D}_0^{\ 2}}}$$

$$\text{C}_2=\frac{n_1}{n_2}\text{D}_0\text{C}_0y$$

$$\frac{\text{F}}{\sigma}=n_1\text{C}_0\sqrt{\frac{1}{\text{D}_0^{\ 2}}+2y\left(x_2\text{Sin}\,\phi+\frac{\text{Cos}\,\phi}{\tau_2}\right)+y^2\left(x_2^{\ 2}+\frac{1}{\tau_2^{\ 2}}\right)}$$

$$\text{E}_2=\frac{n_2}{n_1}\frac{\text{E}_1}{\text{D}_0\left(1+x_1\text{Cos}\delta+\frac{\text{Sin}\delta}{\tau_1}\right)}$$

also

$$\text{Iron loss}=\text{P}_0\text{Sin}\delta\left\{\frac{1}{\text{D}_0^{\ 2}}+2y\left(x_2\text{Sin}\phi+\frac{\text{Cos}\phi}{\tau_2}\right)+y^2\left(x_2^{\ 2}+\frac{1}{\tau_2^{\ 2}}\right)\right\}$$

$$\text{Copper losses}=\text{P}_0\left\{y^2\text{D}_0^{\ 2}\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}+2\frac{x_2}{\tau_1}\text{Cos}\delta+2\frac{\text{Sin}\delta}{\tau_1\tau_2}\right)+2y\frac{\text{Sin}(\delta+\phi)}{\tau_1}\right.$$
$$\left.+\frac{1}{\text{D}_0^{\ 2}\tau_1}\right\}$$

$$\tan\alpha=\frac{\text{Cos}\delta+y\text{Sin}\phi+(\text{X}+2p\text{Sin}\phi)y^2}{\text{Sin}\delta+y\text{Cos}\phi+(\text{T}+2p\text{Cos}\phi)y^2}\ \text{(q.p.)}$$

$$\text{Cot}\beta=\tan(\delta+\phi)+\frac{y}{\text{Cos}(\delta+\phi)}\ \text{(q.p.)}$$

$$\tan\lambda = \frac{\text{Sin}\phi + Xy}{\text{Cos}\phi + Ty} \text{ (q.p.)}$$

where a and $\pi+\lambda$ are the angles that $\bar{C}_1$ and $\bar{C}_2$ are behind $\bar{E}_1$ in phase respectively, and $\pi-\beta$ is the angle $\bar{C}_2$ is behind $\bar{C}_1$.

If $\pi+\epsilon$ be the angle $\bar{E}_2$ is behind $\bar{E}_1$ in phase, then

$$\epsilon = \lambda - \phi$$

and $\tan\epsilon = \dfrac{qy}{1+py}$

a small quantity of the first order, so that $\bar{E}_2$ and $\bar{E}_1$ are always approximately in opposite phases.

Obviously all the above formulae will apply to non-inductive loads when ϕ is made zero in them.

20. The pressure drop at the secondary terminals from no load to any value of the load can now be expressed in terms of the load, its power factor, the transformer numerics and the leakage coefficients as follows :—

We have (see § 19)

$$E_2 = \frac{n_2}{n_1} \frac{E_1}{D_0\left(1 + x_1\text{Cos}\delta + \frac{\text{Sin}\delta}{\tau_1}\right)}$$

so that

$$\frac{E_2 \text{ (at load given by } y)}{E_2 \text{ (at no load)}} = \frac{1}{D_0}$$

$$= \frac{1}{1 + py + \frac{1}{2}(4p^2 + q^2)y^2}$$

$$= 1 - py - \tfrac{1}{2}(2p^2 + q^2)y^2$$

and the percentage drop for any load given by y

$$= 100y[p + \tfrac{1}{2}(2p^2 + q^2)y].$$

Remembering the values of p and q (§ 18) we see that the drop depends on the sum of the reciprocals of the transformer numerics (*i e.* on T) and on the sum of the leakage coefficients. We also see that for non-inductive loads the leakage effect on the drop is only a second-order term, while for inductive loads it is a first-order term, thus showing how important it is to have a small leakage in a transformer that has to operate inductive loads.

21. If the transformer were so designed that at full load it works with maximum efficiency, then the full load value of y or

$$\frac{P_2}{P_0 \mathrm{Cos}\phi} = \sqrt{\frac{\mathrm{Sin}\delta}{T}}\left\{1 - 2p\sqrt{\frac{\mathrm{Sin}\delta}{T}}\right\} \qquad \text{(see §§ 15, 18).}$$

and the percentage drop from no load to full load in such a case is

$$= 100\sqrt{\frac{\mathrm{Sin}\delta}{T}}\left\{p - \left(p^2 - \frac{q^2}{2}\right)\sqrt{\frac{\mathrm{Sin}\delta}{T}}\right\}$$

which shows that the limit of possible excellence in regulation of a transformer, designed as above and perfectly wound, *i.e.*, having no magnetic leakage, is when the percentage drop from no load to full load is

$$100\sqrt{T\mathrm{Sin}\delta}\ \mathrm{Cos}\phi.$$

It has already been shown that the maximum efficiency of a transformer is

$$= 1 - \frac{2}{\mathrm{Cos}\phi}\sqrt{T\mathrm{Sin}\delta} \quad \text{(q.p.)}$$

so that as regards both efficiency and regulation $\frac{1}{T\mathrm{Sin}\delta}$ may be taken as the measure of the excellence of a transformer when the magnetic leakage or nature of the winding is not considered.

In general, for a transformer with negligible leakage, the percentage drop from no load to a load P_2 being

$$100\frac{P_2}{P_0 \mathrm{Cos}\phi}\ T\mathrm{Cos}P$$

as $P_0 = \frac{1}{2}\frac{E_1^2}{r_1 \tau_1}$ (q.p.)

the drop (p.c.) $= 200\frac{r_1 P_2}{E_1^2}\left(1 + \frac{\tau_1}{\tau_2}\right)$

$$= 400\frac{r_1}{E_1^2}P_2 \quad \text{(q.p.)}$$

Reducing to practical units we find that the percentage drop in a transformer cannot be less than

$$200 \times \text{Primary ohms} \times \frac{\text{Output in Watts}}{(\text{Primary virtual volts})^2}$$

In no practical case is the leakage negligible, but with interleaved winding $y^2(x_1 + x_2)^2$ will be very small and $y^2\left\{\frac{1}{\tau_1} + \frac{1}{\tau_2}\right\}^2$ is very small in transformers of 20 K.W. or over, hence for such transformers the drop per cent. from no load to a load P_2

$$=\frac{100P_2}{P_0\mathrm{Cos}\phi}\left\{(x_1+x_2)\,\mathrm{Sin}\phi+\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)\mathrm{Cos}\phi\right\}\ \text{(q.p.)}$$

If P_2 be the capacity of the transformer as usually rated on non-inductive load, then $P_2\mathrm{Cos}\phi$ will be its capacity on an inductive load of power factor $\mathrm{Cos}\phi$, as for this output we get approximately the same secondary current as before. Hence we see from the preceding formula that if $X=T\tan\frac{\psi}{2}$ the regulution for loads whose power factors are less than $\mathrm{Cos}\psi$ is better than that for non-inductive loads.

If the load has capacity then ϕ is negative, and the capacity effect in reducing the drop or even producing a rise in voltage with load can easily be deduced from the general expression for the drop given in § 20.

22. When the percentage drop of a transformer for a non-inductive load P_2 is known, we can by means of the formula in § 20 calculate the sum of its leakage coefficients.

For a non-inductive load—

$$\text{Drop (p.c.)}=100y\left\{T+\left(T^2+\frac{X^2}{2}\right)y\right\}$$

where in this case $y=\frac{P_2}{P_0}$

As P_0 is the power absorbed on open secondary divided by the power factor of the transformer on open secondary it can be determined. It has been shown in § 7 how to practically determine τ_1, and τ_2 can be obtained from τ_1 as follows:—

$$\frac{\tau_1}{\tau_2}=\frac{\dfrac{n_1^2}{r_1}}{\dfrac{n_2^2}{r_2}}\ (\S\ 6).$$

r_1 and r_2 can be measured and

$$\frac{n_1}{n_2}=\frac{\text{Primary volts}}{\text{Secondary volts on open secondary}}\ (\S\ 19).$$

Hence substituting in the above equation the values of y, T, and the drop, X or x_1+x_2 can be found.

It is obvious that we can also by means of the general formula in § 20 determine both x_1+x_2 and $\frac{1}{\tau_1}+\frac{1}{\tau_2}$ for any transformer from obsèrvations of the voltage drop for loads with different power factors. This would not be a satisfactory method for determining the transformer numerics, as their values so obtained would depend on the correct reading of small differences.

23. As an illustration of how closely this theory agrees with practice I will discuss the following manufacturer's specification of the performance of a 10 K.W. Westinghouse O.D. transformer.

Primary volts, 2100:— Frequency, 60 periods per sec.:—
Output, 10 K.W.:— Iron loss, 138 watts:—
Copper loss at full load, 159 watts:—

Efficiency (per cent.)		Regulation (per cent.)		
Full load	97.1	Power factor	1.0	1.65
$\frac{3}{4}$ load	97.05	,, ,,	.9	2.45
$\frac{1}{2}$ load	96.55	,, ,,	.8	2.65
$\frac{1}{4}$ load	94.4	,, ,,	.6	2.80

In the first place we will find whether any definite values of X and T will simultaneously satisfy the four equations in these two quantities obtained from the four observations of drop for different power factors.

If we substitute x for yX and t for yT in the general expression in § 20 for the drop per cent. (R say) we get

$$R=100\left\{x\operatorname{Sin}\phi+t\operatorname{Cos}\phi+x^2\left(\operatorname{Sin}^2\phi+\frac{\operatorname{Cos}^2\phi}{2}\right)+xt\operatorname{Sin}\phi\operatorname{Cos}\phi+t^2\left(\operatorname{Cos}^2\phi+\frac{\operatorname{Sin}^2\phi}{2}\right)\right\}$$

By means of this equation x and t were determined from the first two observations of the regulation and found to be

$x=.0218,\quad t=.016.$

Using these values and calculating R for the power factors .8 and .6, we found that when

$\operatorname{Cos}\phi=.8 \quad R=2.66$

$\operatorname{Cos}\phi=.6 \quad R=2.78$

which agree very closely with the observed values of R, namely 2.65 and 2.80.

As in each case the full load is 10Cosϕ K.W. (see § 21) and in the above

$$y=\frac{\text{Full load}}{P_0\operatorname{Cos}\phi}$$

we have at full load

$$y=\frac{10000}{P_0}.10^7$$

but $x=yX=.0218$ and $t=yT=.016$

$$\therefore 10^7\frac{X}{P_0}=\frac{.0218}{10000} \text{ and } 10^7\frac{T}{P_0}=\frac{\cdot 016}{10000} \qquad \text{(I.)}$$

where P_0 has its usual signification (see § 18).

The iron loss is given as 138 watts, and we may consider it as constant without introducing any appreciable error, so that

$$P_0 \text{Sin}\delta=138.10^7$$

hence from (I.)

$$\left.\begin{aligned} X\text{Sin}\delta&=\frac{.0218\times 138}{10,000}=.000301 \\ T\text{Sin}\delta&=\frac{.016\times 138}{10,000}=.000221 \end{aligned}\right\} \qquad \text{(II.)}$$

The maximum efficiency of the transformer when $\text{Cos}\phi=1$ being very approximately

$$=\frac{1}{1+2\sqrt{T\text{Sin}\delta}} \qquad \text{(see § 15)}$$

is $=.971$

which is the same as the observed value at full load.

The copper losses H_1+H_2 when $\phi=0$ being

$$=P_0\left\{y^2D_0^2\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)+2y\frac{\text{Sin}\delta}{\tau_1}\right\} \qquad \text{(see § 19)}$$

$$H_1+H_2=\frac{T}{P_0}P_2^2\left(D_0^2+\frac{P_0\text{Sin}\delta}{P_2}\right)$$

$$10^7(H_1+H_2)=\frac{.016}{10000}P_2^2\left(D_0^2+\frac{138.10^7}{P_2}\right)$$

where, when $\text{Cos}\phi=1$.

$$D_0^2=1+2\frac{T}{P_0}P_2+\tfrac{1}{2}\left(6\frac{T_2}{P_0^2}+\frac{X^2}{P_0^2}\right)P_2^2 \qquad \text{(see § 18)}$$

and substituting from (I.)

$$D_0^2=1+\frac{32}{10^7}\,\frac{P_2}{10^7}+\frac{1}{10^{11}}\left(\frac{P_2}{10^7}\right)^2$$

Let the load be 10000 z watts so that z is the fraction of full load then

$$P_2=10000z\ 10^7$$

and the copper losses in watts for any fraction z of the full load are

$$=\frac{H_1+H_2}{10^7}=160z^2\left(1+\frac{32}{10^3}z+\frac{1}{10^3}z^2+\frac{138}{10^4z}\right)$$

from which we deduce for the transformer in question that the copper losses are

=167 watts at full load

= 94 watts at $\frac{3}{4}$ load

= 42 watts at $\frac{1}{2}$ load

= 10 watts at $\frac{1}{4}$ load.

The iron loss being 138 watts we find that the efficiency at full load is 97.04 per cent.,

that at $\frac{3}{4}$ load is 97.02 ,, ,,

,, ,, $\frac{1}{2}$ load is 96.54 ,, ,,

,, ,, $\frac{1}{4}$ load is 94.40 ,, ,,

which figures, when compared with those in the maker's specification given above, show a very remarkable agreement.

Thus we have been able to deduce with considerable accuracy from two observations of the regulation for different power factors, and the observed iron loss, the other details of the transformer given by the manufacturer.

If we assume $\delta=50°$ which would mean that the power factor on open secondary was equal to Sinδ or .766, and take $\tau_1=\tau_2$ then $\tau_1=6930$ for this 10 K.W. 60 period transformer; and $x_1+x_2=$ .0004.

24. As a second illustration of the agreement between the foregoing theory and practice I will consider the record of a test of a Westinghouse transformer, published in Fleming's "Alternate Current Transformer," vol. i., pp. 564, 569.

From the no-load readings of C_1, P_1, and E_2 we can determine as has been explained (§§ 7, 19, 22) τ_1, Sinδ, n_1/n_2 and τ_2, while the voltage drop for any load enables us to find x_1+x_2 when τ_1, τ_2 and Sinδ are known (§ 22).

These constants together with r_1, r_2 and the primary voltage enable us to calculate all the variable quantities connected with the transformer for any load. This has been done for the above transformer tested by Fleming, and the calculated values of C_1, C_2, P_1, η, and Cosα for each output in his test are given in Table I. in parallel columns with the values experimentally obtained by him for these quantities.

As the whole behaviour of the transformer is to be evolved from the no-load readings for C_1, P_1, and E_2, and the full load reading of E_2, it is necessary that these should be obtained with accuracy.

The figures for C_1 and P_1 given in the record of the test were probably obtained very near the zeros of the scales of the ammeter and wattmeter used, so instead of relying on single readings I obtained the no-load values of C_1 and P_1 by plotting a few of the readings for them near no-load against the output, and taking the values given by the points where the curves obtained intersect the no-load axis.

In this way I find that C_1 at no-load = .058 amp. and P_1 = 110 watts, which values give the same power factor, .79, as Fleming obtained.

It will be seen on inspection of the following table, that the agreement between the values I have calculated and those observed by Fleming is remarkably close. A very slight modification or correction of the primary wattmeter readings, which the recorded values of the power factor seem to suggest, would make the agreement almost perfect when allowance is made for the inevitable variations from mathematical accuracy of any series of observations.

TABLE I.

Comparison of the measured values of the variables obtained in a test by Fleming of a Westinghouse transformer with values theoretically calculated by the author from no-load values and voltage drop.

Power, 6500 watts. Frequency, 82.5 periods per second.

$r_1 = 5.95$ ohms ; $r_2 = 0.0108$ ohms, at 96°F.

$$E_1 = 2400 \text{ volts}; \; E_2 = \begin{cases} 101 \text{ volts at no-load} \\ 98.6 \text{ volts at 6384 watts.} \end{cases}$$

$$\left\{ \begin{array}{l} \tau_1 = 6950, \; \tau_2 = 6780, \; \text{Sin}\delta = 0.79, \; \dfrac{n_1}{n_2} = 23.76, \\ x_1 + x_2 = 0.003 \end{array} \right\}$$

Output P_2	Primary Current C_1. Obsvd. (Fleming)	Primary Current C_1. Calcd.	Secondary Current C_2. Obsvd. (Fleming)	Secondary Current C_2. Calcd.	Iron Loss Calculated.	Primary Watts P_1. Obsvd. (Fleming)	Primary Watts P_1. Calcd.	Efficiency in Per Cent. Test (Fleming)	Efficiency in Per Cent. Calcd.	Power Factor Cos a. Test (Fleming)	Power Factor Cos a. Calcd.
0	0.050	0.058	0	0	110.0	95	110	0	0	0.79	.79
101	0.100	0.095	1.00	1.00	110.0	205	211	49.3	47.9	0.85	.899
200	0.140	0.134	1.98	1.98	110.0	306	310	65.4	64.5	0.91	.951
296	0.180	0.173	2.94	2.93	109.9	401	406	73.7	72.9	0.93	.978
390	0.218	0.212	3.87	3.86	109.9	493	500	79.1	78.0	0.94	.983
482	0.250	0.250	4.79	4.78	109.9	597	592	80.7	81.4	0.99	.987
806	0.382	0.383	8.00	7.99	109.8	920	917	87.6	87.9	1.00	.998
1019	0.472	0.472	10.15	10.11	109.7	1139	1131	89.5	90.1	1.00	.998
1311	0.580	0.595	13.07	13.02	109.6	1440	1425	91.1	91.0	1.03	.998
1802	0.800	0.801	18.00	17.92	109.5	1930	1919	93.4	93.9	1.00	.998
1992	0.880	0.881	19.90	19.82	109.4	2118	2110	94.1	94.4	1.00	.998
2193	0.960	0.966	21.93	21.83	109.3	2330	2313	94.1	94.8	1.01	.998
2474	1.080	1.085	24.74	24.65	109.2	2609	2597	94.8	95.2	1.01	.997
2966	1.285	1.293	29.66	29.61	109.0	3096	3094	95.8	95.9	1.00	.997
3713	1.610	1.612	37.20	37.17	108.8	3870	3852	96.0	96.4	1.00	.996
4179	1.810	1.819	42.00	42.10	108.6	4324	4344	96.7	96.6	0.99	.995
4633	2.002	2.006	46.65	46.55	108.3	4792	4789	96.9	96.7	1.00	.995
5090	2.160	2.160	50.40	50.33	108.1	5174	5163	96.7	96.8	1.00	.994
5164	2.240	2.237	52.16	52.01	108.0	5422	5331	95.3	96.9	1.01	.993
5499	2.383	2.382	55.60	55.47	107.8	5702	5674	96.1	96.9	1.00	.993
5700	2.478	2.471	57.68	57.57	107.7	5885	5880	96.9	96.9	0.99	.992
5867	2.550	2.544	59.32	59.31	107.6	6041	6051	97.3	96.9	0.99	.991
6053	2.633	2.625	61.32	61.24	107.5	6271	6242	96.7	97.0	0.99	.991
6142	2.672	2.664	62.16	62.16	107.4	6344	6333	96.8	97.0	0.99	.991
6218	2.700	2.698	63.00	62.96	107.4	6426	6411	96.8	97.0	0.99	.990
6317	2.750	2.741	64.00	64.00	107.3	6522	6513	96.9	97.0	0.99	.990
6384	2.775	2.771	64.74	64.70	107.3	6598	6582	96.9	97.0	1.00	.990

25. The method by which magnetic leakage has been dealt with in the preceding theory is not applicable to open circuit transformers. It will be seen in Section III. that this method depends on the fact that in closed circuit transformers the vector $\overline{n_1C_1+n_2C_2}$ which represents the magnetizing ampere turns is small compared with either $\overline{n_1C_1}$ or $\overline{n_2C_2}$, throughout the greater portion of the working range; or, differently stated, that C_1 and C_2 are practically in opposition in phase, and that $n_1C_1-n_2C_2$ is small relatively to either n_1C_1 or n_2C_2 from a small fraction of full load onwards, in closed-circuit transformers.

In open-circuit transformers, on account of the great reluctance of their magnetic circuits, the magnetizing ampere turns are necessarily high, and neither of the two conditions stated above are approximately fulfilled unless over a small range near full load.

If we neglect leakage, or be satisfied with the rough approximation to its effects that the present method affords for open-circuit transformers, then the theory given—as it is equally valid in other particulars for the two types—will apply with fair approximation to accuracy to the open-circuit type, especially in the neighbourhood of full load.

There will be considerable difference, however, in the values of the constants and other characteristics of transformers of equal capacity of the two types.

Let us assume that we have two transformers, one of each type, in which the cores, of the same iron, have the same cross section and volume. Let τ_1, τ_2, δ, σ, θ refer to the closed, and τ_1^0, τ_2^0, δ^0, σ^0, θ^0 refer to the open circuit one. Then if they are so wound that when working under similar conditions their resultant fluxes F and F^0 are equal,

(*a*) Since $\frac{F}{\sigma} = \text{amp.}(\overline{n_1C_1 + n_2C_2})$

(where $\sigma = 4\pi$ permeance of magnetic circuit) their *magnetizing ampere turns will be inversely proportional to the permeances of their magnetic circuits.*

(*b*) Their iron losses will be equal or

$$\tfrac{1}{2}w\frac{F^2}{\sigma}\text{Sin}\delta = \tfrac{1}{2}w\frac{F^{02}}{\sigma^0}\text{Sin}\delta^0 \text{ (see § 14)}$$

and as $F = F^0$

$$\frac{\text{Sin}\delta}{\sigma} = \frac{\text{Sin}\delta^0}{\sigma^0}$$

or *the sines of the angles of magnetic lag of the two transformers are proportional to the permeances of their magnetic circuits.*

Again if the two cores, carrying equal fluxes, have their secondary coils such that the outputs and secondary voltages are equal, the sections of their secondary wires and the numbers of their secondary turns will be equal, and hence the conductivities of their secondary copper belts will be equal, so that

$\frac{\tau_2}{\sigma}=\frac{\tau_2^0}{\sigma^0}$ (see § 6)

or *the secondary numerics are proportional to the permeances of the two magnetic circuits.*

If the closed-circuit transformer be of the core type, so that its windings are similarly arranged to those of the other, then approximately,

$\frac{\tau_1}{\sigma}=\frac{\tau_1^0}{\sigma^0}$ and hence $\mathrm{T}\,\mathrm{Sin}\delta=\mathrm{T}^0\mathrm{Sin}\delta^0$.

If the method of treating leakage was equally legitimate for the two types, we have also, approximately,

$\frac{x_1}{x_1^0}=\frac{x_2}{x_2^0}=\frac{\sigma^0}{\sigma}$ (see § 34), provided the windings are similar.

For a non-inductive load,

$$\theta=\frac{\mathrm{P}_2}{\mathrm{P}_0}=\frac{\mathrm{P}_2\mathrm{Sin}\delta}{\text{Iron loss}}$$

to the first order for both, hence if θ and θ^0 refer to the same output, that is to the same fraction of full load in each

$$\frac{\theta}{\theta^0}=\frac{\mathrm{Sin}\delta}{\mathrm{Sin}\delta^0}=\frac{\sigma}{\sigma^0} \text{ approximately}$$

that is, *the values of the co-ordinate θ for the same output are proportional to the permeances of the magnetic circuits.*

26. In order to compare the rates of approach to opposition of C_1 and C_2 in transformers of the two types, let us consider the equation

$$\mathrm{Cot}\beta=\tan(\delta+\phi)+\frac{y}{\mathrm{Cos}(\delta+\phi)} \quad \text{(see § 19)}$$

which is correct to the first order for both, where $\pi-\beta$ is the angle C_2 is behind C_1 in phase.

For a non-inductive load

$$y=\frac{\mathrm{P}_2}{\mathrm{P}_0}=\frac{\mathrm{P}_2\mathrm{Sin}\delta}{\text{Iron loss}}=\frac{\mathrm{P}_2}{\mathrm{H}_3}\mathrm{Sin}\delta$$

so that

$$\mathrm{Cot}\beta=\tan\delta\left(1+\frac{\mathrm{P}_2}{\mathrm{H}_3}\right)$$

for both types when $\phi=0$.

But δ for the closed circuit type will be 50° or over, and for the open circuit (hedgehog) type will be about 4°; and if we assume (which will be sufficiently accurate for our present purpose) that in transformers of equal capacity of the two types the iron losses are equal, we find, taking the figures for the 10 K.W. transformer discussed in § 23, that β will be given by the equation

$$\text{Cot}\beta = \tan\delta\left(1 + z\frac{10{,}000}{138}\right)$$

where z is the fraction of full load and $\delta = 50°$ for the closed-circuit, and $= 4°$ for the open-circuit transformer.

The figures in the following table, calculated from the above formula, show the relative approach to opposition of C_1 and C_2 in 10 K.W. transformers of the two types.

Fractions of Full Load - - -	0	0.1	0.25	0.5	1.0
β for closed-circuit transformer -	40°	5°	2.5°	1.3°	0.7°
β for open-current transformer -	86°	60°	37°	21°	11°

In both cases the approach to opposition will be quicker for larger transformers, as in them the iron loss is a smaller fraction of the full load output.

For inductive loads having a constant power factor $\text{Cos}\phi$

$$\text{Cot}\beta = \tan(\delta + \phi) + \frac{P_2\text{Sin}\delta}{H_3\text{Cos}\phi\text{Cos}(\delta + \phi)}$$

so that at no-load $\beta = \frac{\pi}{2} - (\delta + \phi)$ or C_1 and C_2 at the beginning of the range are nearer to opposition for both types than when the load is non-inductive, and as $\text{Cos}\phi\text{Cos}(\delta + \phi)$ is less than unity, the successive increments to $\text{Cot}\beta$ for definite fractions of the load will be greater; hence for both types of transformers the approach to opposition of C_1 and C_2 will be more marked with inductive than with non-inductive loads.

Section II.

27. The theory developed in Section I. is easily applicable to the design of a closed-circuit transformer, when the full load, power factor of the load, periodicity, and e.m.f.'s are given.

In the first place we would select the form of the magnetic circuit, and after consideration of the probable cooling surface, volume, and method of cooling to be adopted, decide on the permissible copper and iron losses per unit volume.

If K be the copper loss decided on, per second, per unit volume, at full load, then

$K = \frac{1}{2}\rho c^2$,

whence c, the amplitude of the full-load current density, is known, ρ being the specific resistance of copper at the expected working temperature.

When the iron loss per second, per unit volume, (I. say) is given, the corresponding retardation δ, permeability and flux density can be got from curves similar to those in Fig. I., that have been obtained from a sample of the iron to be used with (*q.p.*) sine wave magnetising currents whose period was the given one.

If γ be the flux density (= B the abscissae in Fig. I.), then we should have between these quantities the relation,

$$\frac{w\gamma^2 \text{Sin}\delta}{8\pi\mu} = \text{I. (see § 1).}$$

Once the form of the magnetic circuit has been selected, its dimensions can be completely specified by two variables. The output at full load, P_2 say, can be expressed in terms of these two variables, for P'_2, the power passed down to the secondary and developed in it, is given by the equation,

$$P'_2 = \tfrac{1}{2} w n_2 C_2 F \text{Cos}\phi' \qquad \text{(I.)}$$

and F = iron section × permissible flux density (γ),

$n_2 C_2$ = total copper section × permissible current density at full load (c).

P'_2 can be obtained from P_2 the given output, and $\text{Cos}\phi'$ from $\text{Cos}\phi$, the given power factor of the load, by the equations,

$$P'_2 = \left\{1 + \frac{\theta_z}{\tau}\right\} P_2$$

and $$\tan\phi' = \tan\phi + \left\{x_2 - \frac{\tan\phi}{\tau}\right\}\theta_z$$

or $$\text{Cos}\phi' = \text{Cos}\phi\left\{1 - \left(x_2 - \frac{\tan\phi}{\tau}\right)\text{Sin}\phi\text{Cos}\phi\,\theta_z\right\}$$

where θ_z is the full load value of θ, when approximate values of the transformer numeric and of the secondary leakage coefficient are known.

[In future τ will be used for either τ_1 or τ_2 when approximate values only are required].

The approximate value of τ required may be found by a rough preliminary calculation, or from a formula such as that given in § 55, when τ for some other transformer of the same type is known, and θ_z, the full load value of ϕ, taken as given by the equation (see § 17),

$$\theta_z \text{Cos}\phi = \sqrt{\frac{z\text{Sin}\delta}{\text{T}}} = \sqrt{\frac{z\tau\text{Sin}\delta}{2}}$$

where z is the chosen ratio of copper to iron losses at full load.

In Section III. of this paper will be shown how to determine the leakage coefficients when the form of the magnetic circuit, method of winding, and space factors, are known. As these details will be decided on in the first place, a fairly accurate preliminary value of x_2 can be obtained, and hence the value of $\text{Cos}\phi'$ by means of either of the equations given above.

Thus we can obtain from equation I. above the product of the section of the iron circuit by the total section of the secondary copper circuit.

Obviously, as a first approximation we might in equation I. consider $P'_2 = P_2$ and $\text{Cos}\phi' = \text{Cos}\phi$, which would amount to neglecting secondary leakage and secondary copper loss.

A second relation between the two variables is obtained by expressing the condition that, at full load, the ratio of the copper to the iron losses is to have a definite chosen value z.

If z be chosen as unity for a transformer that is to operate a non-inductive load, or as

$$1 - \text{Sin}\phi\left\{x_2\text{Sin}\delta - \frac{\text{Cos}\delta}{\tau}\right\}\sqrt{\frac{2\tau}{\text{Sin}\delta}}$$

(using the approximate value of τ) for one that is to operate an inductive load whose power factor is $\text{Cos}\phi$, then in either case full load would correspond with maximum efficiency (see § 16).

As the efficiency of a transformer keeps very near its maximum value for a wide range on either side of the maximum position, it is not a matter of great importance to arrange that maximum efficiency exactly corresponds to full load.

As copper costs more than iron it may be more economical to use a relatively smaller quantity of copper, and put up with a larger copper loss than when $z = 1$.

From these two relations the two variables, and hence the dimensions of the transformer, are determined, and the formulae in Section I. enable us to arrive at the different details such as n_1, n_2, a_1, a_2, r_1, r_2, τ_1, τ_2, etc., when the e.m.f.'s on the primary and secondary sides are given.

28. As an illustration of this method I will work out the theoretical* design of transformer, to transform from 2200 to 220 virtual volts at 50 periods, and to carry an inductive load of 10 K.W. whose power factor is .8.

Selecting the shell type of transformer, one of the laminae of which is shaped as in Fig. 5, we will suppose the iron tongue to be of square cross section (2β, 2β) and the windows or winding apertures to be also square ($2b$, $2b$).†

Hence the mean length of the magnetic circuit is $4\ (2b+\beta)$, and the mean length of a turn of either primary or secondary coil (so wound as to be the same for both) is $8\ (b+\beta)$.

If p‡ the space factor of the iron be taken $=.9$, then the cross section of the iron circuit is $=4p\beta^2=3.6\beta^2$, and the volume of the iron

$=16p\beta^2(2b+\beta)=14.4\beta^2(2b+\beta)$.

The space factor p will not only enable us to allow for insulation between the laminae, but also for ventilating or cooling ducts, if such are deemed necessary.

Let us decide that the iron loss shall be 10^5 ergs. per second, per unit volume.

With a sample of transformer iron .045 cm. thick I have found with my wave tracer,‖ when the iron loss per cm.^{3} per second was 10^5 ergs. at 50 periods, the magnetising current wave form being slightly peaked, that

$\gamma=4800$, $\delta=52^\circ$ $\mu=2250$ (q.p.)

so for the present design I will assume, for the iron to be used,

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* Called theoretical because the details are fully worked out in accordance with the theory already given. A knowledge of the theory and experience will, however, enable one to make sufficiently accurate allowance for most of the small correcting terms, instead of having to calculate them in each particular case.

† This is far from being the most efficient shape, as will be shown in Section III.

‡ The different factors and constants assumed are not given with any authority. The purpose of this part of the paper is merely illustrative.

‖ Phil. Mag., Nov., 1903.

that

$\gamma = 4847$, $\delta = 50°$ $\mu = 2250$,

which satisfy the sine wave equation

$$\frac{w\gamma^2 \text{Sin}\delta}{8\pi\mu} = 10^5$$

which states that the iron loss per cm.^{3} per second is 10^5 ergs.

The core loss will therefore be $144.10^4\beta^2(2b+\beta)$ at full load.

29. The kind of winding to be adopted will depend on the excellence of regulation for inductive loads that is required. In Section III. will be shown how to calculate the leakage coefficients for different kinds of windings, and how the regulation for inductive loads that these windings will give, may with considerable accuracy be predetermined.

A very important consideration with regard to the arrangement of the windings is clearly shown by the general expression for the efficiency given in § 15. It will be seen that in the denominator, positive terms depending on x_2, the leakage coefficient of the secondary, occur, of the first order in small quantities for inductive loads, and of the second order for non-inductive loads. Hence it is obvious that if we have a choice, we should so place the secondary windings or sections that x_2 is the least possible. Now it will be shown in Section III. that, with interleaved windings symmetrically arranged with regard to the middle line across the window, which line must therefore bisect the central section of one of the coils, the leakage coefficient of that coil to which the two outer sections belong is negative. Hence the most efficient arrangement is that in which the two outer sections belong to the secondary or output coil. The regulation will be the same whether one or other of the coils has the outer positions, as it depends on $x_1 + x_2$, which is little or not at all affected by the interchange.

Assuming that for the present design a winding in three sections will give satisfactory regulation, we will place the whole primary coil as a single section in the central position between the two halves of the secondary coil.

For such a winding, when the iron and copper losses per cm^3, and the space factors are as we assume in this design, and when $\mu = 2250$ (see § 51),

$x_1 = .00129 \quad x_2 = -.00024$
$x_1 + x_2 = .00105.$

30. If K_1 and K_2 be the permissible copper losses per second per $cm.^3$ at full load, K_1 for the primary being decided on as say 15.10^4 ergs, and K_2 differing from K_1 by a small amount which will depend on how the copper losses are to be divided between the two coils; then

$$K_1 = \tfrac{1}{2}\rho c_1^2, \quad K_2 = \tfrac{1}{2}\rho c_2^2,$$

where c_1 and c_2 are the amplitudes of the current densities, and ρ the specific resistance of copper at the expected working temperature.

Let us take $\rho = 1800$ abs.

then $\qquad c_1 = 12.91$ abs.

Let s_1, s_2, be the sectional areas, l_1, l_2 the mean turns, and q_1, q_2 the space factors of the two coils; so that $q_1 s_1$, $q_2 s_2$, are their total copper sections, and $q_1 s_1 l_1$, $q_2 s_2 l_2$, their total copper volumes; then (see § 6) since $l_1 = l_2$,

$$\frac{q_1 s_1}{q_2 s_2} = \frac{\tau_1}{\tau_2} = 1 + \kappa \text{ say,}$$

where κ is a small quantity to be determined, depending on the current densities in the two coils at full load.

This equation, together with $s_1 + s_2 = 4b^2$, give us the copper sections,

$$q_1 s_1 = 2Qb^2\left\{1 + \frac{Q}{2q_2}\kappa\right\}$$

$$q_2 s_2 = 2Qb^2\left\{1 - \frac{Q}{2q_1}\kappa\right\}$$

where $Q = \dfrac{2q_1q_2}{q_1 + q_2}$ = the harmonic mean of q_1 and q_2.

Let us assume for the copper space factors

$$q_1 = .5, \quad q_2 = .7$$

then $\quad Q = .583$

and $\quad q_1 s_1 = 1.166(1 + .42\kappa)b^2$

$\qquad q_2 s_2 = 1.166(1 - .58\kappa)b^2.$

31. If we arrange so that the current densities in the two coils shall be equal at full load, then

$$c_1 = c_2, \quad K_1 = K_2$$

and

$$\frac{q_1 s_1}{q_2 s_2} = \frac{n_1 C_1}{n_2 C_2} = 1 + \frac{\text{Sin}(\delta + \phi)}{\theta_z \text{Cos}\phi} \text{ (see § 13)}$$

$$\text{or } \kappa = \frac{\text{Sin}(\delta + \phi)}{\theta_z \text{Cos}\phi}$$

where C_1, C_2, and θ_z are full load values.

In addition let us arrange that the copper and iron losses shall be equal at full load. Then (see § 17), as $z = 1$,

$$\theta_z \text{Cos}\phi = \sqrt{\frac{\text{Sin}\delta}{T}} = \sqrt{\frac{\tau \text{Sin}\delta}{2}}$$

so that

$$\kappa = \text{Sin}(\delta + \phi)\sqrt{\frac{2}{\tau \text{Sin}\delta}}.$$

For the determination of κ and other small correcting terms, an approximate value of τ must be known. We can easily obtain one by a rough preliminary calculation in which these correcting terms are neglected, or from the formula given in § 55, when τ for some other transformer of the same type is known.

The first method gives us $\tau = 6000$;

hence as

$\delta = 50°$, $\text{Sin}\delta = .766$,

$\text{Cos}\phi = .8$, $\phi = 37°$,

we find that,

$\theta_z \text{Cos}\phi = 48$, $\theta_z = 60$,

$\kappa = .02$, $\tau_1 / \tau_2 = 1.02$,

$\frac{\theta_z}{\tau} = .01$

and taking $x_2 = -.00024$ (see § 28),

$\text{Cos}\phi' = .808$ at full load (see §§ 10, 27).

Substituting the value of κ thus determined in the expressions for $q_1 s_1$, and $q_2 s_2$ we get

$q_1 s_1 = 1.176 b^2$, $q_2 s_2 = 1.153 b^2$.

Hence, the total volume of copper being

$= l(q_1 s_1 + q_2 s_2) = 8(b + \beta)(q_1 s_1 + q_2 s_2)$,

it is

$= 18.63 b^2 (b + \beta)$,

and as the copper losses at full load are

$= 15.10^4 \times \text{volume}$,

they are

$$=2795.10^3 b^2(b+\beta).$$

32. As we intend that the copper and iron losses shall be equal at full load,

$$\frac{2795 b^2(b+\beta)}{1440\beta^2(2b+\beta)}=1 \text{ (see §§ 28, 31).}$$

or $\dfrac{u+1}{u^2(u+2)}=.5152$

where $u=\beta/b$,

from which equation u (the positive root) can easily be determined by trial (using a slide rule) and is $=1.151$,

hence $\beta=1.151b$.

33. The output P_2, being 10 K.W., that is, 10^{11} ergs. per second, and (see § 13) as

$$P'_2=\left\{1+\frac{\theta_z}{\tau}\right\}P_2,$$

we find, using the approximate value for $\dfrac{\theta_z}{\tau}$ given in § 31, that P'_2, the total power developed in the secondary is

$$=1.01 \,.\, 10^{11},$$

but $P'_2=\frac{1}{2}E'_2C_2\text{Cos}\phi'$

$$=\tfrac{1}{2}wn_2FC_2\text{Cos}\phi';$$

hence, using the value of $\text{Cos}\phi'$ given in § 31, and remembering that $w=2\pi \,.\, 50=100\pi$,

$$n_2C_2F=\frac{202 \,.\, 10^9}{100\pi \,.\, .808},$$

$$=796 \,.\, 10^6.$$

Now c_2 being the permissible current density $=12.91$ (see § 30)

$$n_2C_2=c_2q_2s_2=12.91 \,.\, 1.153b^2$$

$$=14.9b^2$$

and γ being the permissible flux density $=4847$,

$$F=4p\beta^2\gamma=3.6 \,.\, 4847 \,.\, \beta^2,$$

$$=17450\beta^2.$$

Hence $b^2\beta^2=\dfrac{796 \,.\, 10^6}{14.9 \,.\, 17450}=3062$

and as $\beta=1{\cdot}151b$

we find that $\beta=7.98$ and $b=6.93$,

which determine the carcass of the transformer.

Substituting these values of β and b in the expressions for the copper and iron losses at full load given in §§ 31, 28, we find that each is equal to $200.3 \cdot 10^7$, that is to 200.3 watts.

Hence the efficiency at full (inductive) load of 10 K.W. will be

$$\frac{10000}{10400.6} \text{ or } 96.15 \text{ per cent.}$$

34. The numerics τ_1 and τ_2 can now be determined for as

$\tau_1 = 4\pi w \times$ Conductance of primary copper belt $\times$ Permeance of magnetic circuit

$$\tau_1 = 4\pi w \frac{q_1 s_1}{8(b+\beta)\rho} \frac{4p\beta^2\mu}{4(2b+\beta)}$$

$$= \frac{400 \cdot \pi^2 \cdot 1.176 \cdot 48.08 \cdot 3.6 \cdot 63.7 \cdot 2250}{8 \cdot 14.91 \cdot 1800 \cdot 4 \cdot 21.84},$$

$$= 6140,$$

and as $\tau_1 = 1.02\tau_2$,

$$\tau_2 = 6020.$$

From the results already obtained the accurate full load value of $\theta_z \mathrm{Cos}\phi$ can now be calculated by means of the formula in § 17, remembering that $z=1$.

Thus we get at full load

$$\theta_z \mathrm{Cos}\phi = 47.75.$$

35. To determine n_1 the number of primary turns, we have (see § 13),

$$w n_1 \mathrm{F} = \frac{\mathrm{M}}{\mathrm{D}} \mathrm{E}_1,$$

where M and D are the full load values of the expressions for them given in § 12. These can now be obtained as all the quantities required for their calculation are known.

Thus $\mathrm{M}=1$, $\mathrm{D}=1.044$,

and as

$\mathrm{F} = 17450\beta^2 = 1111.10^4$ (see 33),

$\mathrm{E}_1 = 2200 \cdot 1.414 \cdot 10^8$,

$w = 100\pi$,

we find that $n_1 = 854$.

36. To determine n_2 the number of secondary turns, so that at full load the secondary e.m.f. shall be 220 volts.

From § 13,

$$E_2 = \frac{n_2}{n_1}\frac{E_1}{D}$$

$$\text{or } n_2 = \frac{n_1 D}{10}$$

and as D=1.044 (see § 35),

$n_2 = 89.16$.

37. The sectional area of the conductors to be used being

$$\frac{q_1 s_1}{n_1} \text{ and } \frac{q_2 s_2}{n_2},$$

for the primary and secondary respectively, we find (see §§ 31, 33),

Sect. area of primary conductor $=0.0662 cm^2$,

Sect. area of secondary conductor $=0.6218 cm^2$.

The mean length of a turn of either coil being

$=8(b+\beta)=119.3$

and as $\rho=1800$,

the resistances of the coils (warm) are

Primary, 2.77 ohms.

Secondary, 0.0308 ohms.

38. The terminal voltage at no load being (see § 13)

$$\frac{n_2}{n_1}\frac{E_1}{1+x_1\operatorname{Cos}\delta+\frac{\operatorname{Sin}\delta}{\tau_1}}$$

is =229.5 volts,

and as it is 220 volts at full inductive load the drop will be 4.14 per cent.

If the same transformer operated a non-inductive load it would be rated as of

$$\frac{10}{.8} = 12.5 \text{ K.W. capacity,}$$

and its voltage drop from no load to a non-inductive load of 12.5 K.W. would be 1.5 per cent.

39. By means of the formulae obtained in Section I. the curves shown in Fig. 3 and Fig. 4 were constructed for the

transformer we have designed. Those in Fig. 3 refer to it when operating the kind of load for which it was designed, namely, one with a constant power factor of .8, while those in Fig. 4 refer to the same transformer when operating a non-inductive load. A comparison of these theoretical curves with similar ones obtained practically from actual transformers will afford a further illustration of the agreement between the theory I have given and practice.

40. In the preceding design it was arranged that the current densities in the two coils should be equal at full load. Any other desired distribution of current density, however, can be equally well dealt with.

Thus if K_1 and K_2, the copper losses per cm^3 at full load, be each given, we know c_1 and c_2 as $K = \frac{1}{2}\rho c^2$, and

$$\frac{q_1 s_1 c_1}{q_2 s_2 c_2} = \frac{n_1 C_1}{n_2 C_2} = 1 + \kappa$$

κ is determined as in § 30, by aid of an approximate value of τ, so that the ratio

$\frac{q_1 s_1}{q_2 s_2}$ can be found ;

and as $s_1 + s_2 = 4b^2$ we can determine s_1 and s_2 and proceed as before.

Again if K_1 be given, and we wish to arrange so that at full load the primary and secondary copper losses shall be equal, we have

$$\frac{q_1 s_1 c_1}{q_2 s_2 c_2} = 1 + \kappa \qquad \text{(as above),}$$

and $q_1 s_1 l_1 K_1 = q_2 s_2 l_2 K_2$

or $\quad q_1 s_1 c_1^2 = q_2 s_9 c_2^2$

as $l_1 = l_2$ and $K = \frac{1}{2}\rho c^2$

hence $\frac{q_1 s_1}{q_2 s_2} = (1 + \kappa)^2$,

which with $s_1 + s_2 = 4b^2$

determine s_1 and s_2 and we proceed as before.

In transformers of the core or H type, in which the primary coil almost completely surrounds the secondary, l_1 is greater than l_2, and the preceding method would have to be modified.

When s_1 and s_2 have been determined as before, l_1 and l_2 can be expressed in terms of the two variables b and β, and the ratio of the losses

$$\frac{q_1 s_1 l_1 K_1 + q_2 s_2 l_2 K_2}{I \text{ (volume of iron)}}$$

being equated to the selected value z gives us an equation, slightly more complex than that for a shell transformer, for determining β/b and the rest follows as in the preceding cases.

41. When we select for the section of the iron tongue and for the windows or winding apertures different shapes from that selected in § 28, the method of procedure is fairly obvious. In general if 2β, $2\beta'$, be the section of the tongue, $2\beta'$ being measured perpendicular to the planes of the laminae ; and if $2b$, $2b'$, be the dimensions of the window $2b'$ being measured parallel to the tongue;

the volume of iron $=16p\beta\beta'(b+b'+\beta)$

and the volume of copper $=16Qbb'(\beta+\beta'+2b)$

[neglecting the small correcting terms in κ depending on the distribution of current densities in the two coils], where p and Q have the same signification as before, and if the iron and copper losses are to be equal at full load

$$\frac{bb'(\beta+\beta'+2b)}{\beta\beta'(b+b'+\beta)}=\frac{pI}{QK}=\frac{.9\ .\ 10^5}{.583\ .\ 15\ .\ 10^4}=1.029$$

if we adopt the same values for the data as before.

The values of β/b for a few special shapes are as follows :—

(*a*) If $b=b'$, $2\beta'=3\beta$,
$\beta/b=.984$.

(*b*) If $b=b'$, $\beta'=2\beta$,
$\beta/b=.886$.

(*c*) If $2b'=3b$, $2\beta'=3\beta$,
$\beta/b=1.141$.

(*d*) If $2b'=3b$, $\beta'=2\beta$,
$\beta/b=1.025$.

(*e*) If $b'=2b$, $2\beta'=5\beta$,
$\beta/b=1.042$.

Let us determine approximate values of τ for transformers of the above shapes whose output on non-inductive load shall be 12.5 K.W. at 50 periods, the normal rating of the transformer already designed.

We have (see § 33)

$\frac{1}{2}wn_2C_2F=12.5 \,.\, 10^{10}+$Secondary copper loss, and the secondary copper loss may in this connection be neglected in a rough determination of τ; but

$$n_2C_2=c_2 \,.\, \frac{Q}{2} \,.\, 4bb'=15.05bb',$$

$$F=\gamma \,.\, p \,.\, 4\beta\beta'=17450\beta\beta',$$

taking the values $c_2=12.91$, $\gamma=4847$ already used; hence

$$bb'\beta\beta'=\frac{12.5 \,.\, 10^{10}}{50\pi \,.\, 15.05 \,.\, 17450}=3030,$$

which with the ratios β/b above enables us to determine β and b in each case.

The formula for the numeric τ can be put in the form

$$\tau=\frac{\pi\mu}{2\rho c\gamma}\frac{\text{Output}}{(\beta+\beta'+2b)\,(b+b'+\beta)}=\frac{3{,}925.000}{(\beta+\beta'+2b)\,(b+b'+\beta)}$$

by means of which its values in the five special cases considered can be determined.

Thus we obtain the following details given in tabular form.

	(*a*)	(*b*)	(*c*)	(*d*)	(*e*)
b	6.76	6.63	5.67	5.57	4.86
b'	6.76	6.63	8.50	8.35	9.72
β	6.65	5.87	6.47	5.71	5.06
β'	9.97	11.74	9.70	11.42	12.65
Volume of copper	12840	12660	12450	12280	12090
Volume of iron	19250	18990	18680	18420	18140
τ	6460	6650	6910	7080	7280

We also find that the iron losses at full load, which are half the total losses, are for (*a*) 192.5, (*b*) 189.9, (*c*) 186.8, (*d*) 184.2, and (*e*) 181.4 watts, so that transformer (*e*) is the most efficient of the series, having an efficiency at full load of 97.2 per cent. Obviously the transformer designed in detail with square windows and square tongue is of a less efficient shape than any of these, as its iron loss at full load is 200.3 watts.

The volume of iron in each of the present series in cub. cms. is 100 times the iron loss in watts. In (*e*) it is 18140 cub. cms., which corresponds to 25 lbs. of iron core for each kilowatt of full load activity.

It is worth noting that, for transformers of the same type made of similar iron, the percentage iron and copper losses and the weights of copper and iron per kilowatt of full load activity, are inversely proportional to the fourth root of the product of the output into the frequency.

Section III.

Transformer Leakage.

42. In addition to the magnetic lines forming the main flux $\overline{F}$, produced by the combined action of $\overline{C_1}$ and $\overline{C_2}$ and looped on both circuits of a transformer, there are other lines, the leakage lines, that are only partially looped on the two coils and that traverse the space occupied by the coils, in some cases completing their circuits in the iron. It will be shown that, after the transformer is somewhat loaded, the effect of these leakage lines on its operation is the same as would be produced by two fluxes ; one, the primary leakage flux, supposed to consist of lines in phase with the primary current that are looped on the primary and miss the secondary circuit, and the other, the secondary leakage flux, suppposed to consist of lines in phase with the secondary current, that are looped on the secondary and miss the primary circuit.

43. Let L_1 and L_2 be the inductances of the two coils when the leakage lines only are considered, M_{12} the mutual inductance of the primary on the secondary, that is the number of leakage lines looped on the secondary arising from unit current in the primary, and M_{21} the mutual inductance of the secondary on the primary. (M_{12} will not in general be equal to M_{21}).

The e.m.f. $\overline{e_1}$ generated in the primary coil by variation of the leakage lines due to $\overline{C_1}$ and $\overline{C_2}$ will be represented by the vector

$$\overline{wL_1C_1 + wM_{21}C_2}$$

after it has been turned through a right angle in the negative direction : but since the vector

$$\overline{n_1C_1 + n_2C_2}$$

which represents the magnetising ampere turns, is, after the transformer is somewhat loaded, negligible in comparison with either $\overline{n_1 C_1}$ or $\overline{n_2 C_2}$ (see §§ 9, 13, 25, 26),

$$\bar{C}_2 = -\frac{n_1}{n_1} C^1$$

and hence $\bar{e}_1$ is the vector

$$w\left\{\frac{L_1}{n_1} - \frac{M_{21}}{n_2}\right\} n_1 \bar{C}_1,$$

after it has been turned back through a right angle and

$$\text{amp. } e_1 = w\left\{\frac{L_1}{n_1} - \frac{M_{21}}{n_2}\right\} n_1 C_1$$

Thus we see that the effect of the leakage lines on the primary circuit is the same, when the transformer carries a load, as would be produced by a flux

$$= \left\{\frac{L_1}{n_1} - \frac{M_{21}}{n_2}\right\} \bar{C}_1$$

looped on it but not on the secondary ; but in § 2 this flux was specified by $x_1 \sigma n_1 \bar{C}_1$
where x_1 is the primary leakage coefficient, and σ is 4π times the permeance of the magnetic circuit, hence

$$x_1 = \frac{1}{n_1 \sigma}\left\{\frac{L_1}{n_1} - \frac{M_{21}}{n_2}\right\}.$$

Similarly if x_2 be the secondary leakage coefficient

$$x_2 = \frac{1}{n_2 \sigma}\left\{\frac{L_2}{n_2} - \frac{M_{12}}{n_1}\right\}.$$

44. Let us determine L_1, L_2, M_{12}, M_{21} and thence x_1 and x_2 for a shell transformer in which both primary and secondary coils are single.

This must be done in two parts. We must determine the values of those portions of the above coefficients that are due to the leakage lines that cross the windows, as well as the values of their remaining portions that are due to the leakage lines that cross those parts of the coils that are not embedded in the iron. Let L'_1, L'_2, M'_{12}, M'_{21}, x'_1, x'_2 be the former, and L''_1, L''_2, etc., the latter portions of the above coefficients.

Let 2β be the width of the iron tongue measured in the plane of one of the laminae from window to window, and let $2\beta'$ be its height measured perpendicular to the laminae. Also (see Fig. 5) let PP′ the breadth of the window=D, PO the thickness of the primary coil $=b_1$, and SO the thickness of the secondary coil $=b_2$.

In order to determine L'_1, L'_2, etc., consider first the leakage lines *produced by* C_1 that cross the windows through the spaces occupied by the primary coil.

If we draw two planes A′A, B′B, Fig. 5, the same distance z on either side of the median plane of the primary, the magnetic forces due to the two portions of the primary coil AP and BO neutralize each other within the space AA′BB′, so that the M.M.F. round the circuit A′ABB′ is due to the portion of the primary it encloses and is

$$=\frac{8\pi n_1 C_1}{b_1}z,$$

and the flux circulating in this circuit through the elemental rectangle $2\beta' dz$ at z is

$$=\frac{8\pi\beta'}{Db_1}n_1 C_1 z dz$$

as it crosses the window twice.

This flux encircles the current

$$\frac{n_1 C_1}{b_1}2z$$

and as the energy associated with naturally looped flux and current is $\frac{1}{2}$ flux × current, the energy of C_1 due to the lines through the space occupied by the primary coil is

$$=\frac{8\pi\beta'}{Db_1^2}n_1^2 C_1^2 \int_{0'}^{\frac{b_1}{2}} z^2 dz$$

$$=\frac{\pi}{6}\ \frac{2\beta' b_1}{D}n_1^2 C_1^2 \qquad \text{(I.)}$$

Thus we see that the energy of C_1 due to the leakage lines that cross and recross the space occupied by the coil carrying C_1 is equal to $\frac{\pi}{6}\ n_1^2 C_1^2 \times$ permeance of this space across the window.

Again the M.M.F. due to C_1, in any circuit that crosses the space occupied by the secondary coil, and completes itself through the iron round the primary coil is uniform, and $=4\pi n_1 C_1$ and sends through the secondary space (section $2\beta' b_2$) the flux

$$\frac{8\pi\beta' n_1 C_1}{D} b_2$$

(neglecting the reluctance of the iron).

This flux encircles all of n_1C_1 and the energy associated with it is

$$\frac{4\pi\beta'}{D} n_1^2 C_1^2 b_2,$$

so that the total energy of C_1 due to the leakage lines it produces and that cross the window is

$$= \frac{4\pi\beta'}{D}\left\{b_2 + \frac{b_1}{12}\right\} n_1^2 C_1^2.$$

The other window contributes an equal amount, hence as the sum is also equal to $\frac{1}{2}L'_1C^2$,

$$\left.\begin{aligned} &L'_1 = \frac{16\pi\beta'}{D}\left\{b_1 + \frac{b_1}{12}\right\} n_1^2. \\ &\text{Similarly,} \\ &L'_2 = \frac{16\pi\beta'}{D}\left\{b_1 + \frac{b_2}{12}\right\} n_2^2. \end{aligned}\right\} \qquad \text{(II.)}$$

The latter of the two fluxes already considered is partially looped on n_2C_2 and the mutual energy of C_1 and C_2 due to this can easily be found as follows.

The flux through $2\beta' dy$ where $y = OE$, Fig. 5, is

$$= 4\pi n_1 C_1 \frac{2\beta' dy}{D},$$

and it is looped on the portion

$$\frac{n_2 C_2}{b_2} y \text{ of } n_2C_2$$

so that the mutual energy is

$$= \frac{8\pi\beta'}{Db_2} n_1 n_2 C_1 C_2 y dy.$$

Integrating from $y = o$ to $y = b_2$ we find that the mutual energy of C_2 and the leakage lines due to C_1 is

$$= \frac{4\pi\beta'}{D} n_1 n_2 C_1 C_2 b_2.$$

Thus we see that the mutual energy of the uniform C_1 flux that goes through the C_2 coil, and n_2C_2 is

$$= \tfrac{1}{2} \text{ . flux . } n_2C_2. \qquad \text{(III.)}$$

or is the same as if all the flux through the space were looped on half the total current in the space, a result that will be made use of in § 45, *b*.

Hence, as the other window contributes an equal amount of energy

$$\left.\begin{aligned} M'_{12}C_1C_2 &= \frac{8\pi\beta'}{D}n_1n_2C_1C_2b_2 \\ M'_{12} &= \frac{8\pi\beta'}{D}n_1n_2b_2. \\ \text{Similarly,} & \\ M'_{21} &= \frac{8\pi\beta'}{D}n_1n_2b_1. \end{aligned}\right\} \quad \text{(IV.)}$$

Substituting from (II.) and (IV.) in the equations for x'_1 and x'_2 in § 43, we get

$$\left.\begin{aligned} x'_1 &= \frac{16\pi\beta'}{\sigma D}\left\{b_2 - \frac{5}{12}b_1\right\} \\ x'_2 &= \frac{16\pi\beta'}{\sigma D}\left\{b_1 - \frac{5}{12}b_2\right\} \\ \text{and} & \\ x'_1 + x'_2 &= \frac{28\pi\beta'}{3\sigma D}\left\{b_1 + b_2\right\} \end{aligned}\right\} \quad \text{(V.)}$$

45. The determination of the coefficients x''_1 and x''_2 due to leakage lines other than those that cross the windows can only be approximate.

A fair approximation can, however, be obtained by assuming that these lines form circuits like *aa'cb'ba*, Fig. 6, of which *ba* is in the iron, as the inner surfaces of the coils bear against the iron tongue, *aa'* and *bb'* are parallel to the plane of separation of the coils and *a'cb'* is a semicircle in the air joining *a'* and *b'*.

Let us consider the lines due to C_1.

If *aa'* and *bb'* are a distance z on either side of the median plane of the primary coil, then the M.M.F. round the circuit *aa'cb'ba* is

$$= \frac{4\pi n_1 C_1}{b_1}2z,$$

which will send through the magnetic circuit at z, whose breadth is dz the flux

$$\frac{4\pi n_1 C_1}{b_1}2z\frac{B\,dz}{2D+\pi z}$$

if B be the mean width perpendicular to dz of this elementary circuit. Now where the coil bears against the iron the width is

2β, and at the outer surface of the coil the width is $2\beta+\pi D$, if we assume that the corners of the coil are quadrants of circles.

Hence $B=2\beta+\frac{\pi}{2}D$.

[This allowance for the corners of the coil may be considered rather large, but if so it is compensated by the fringing of the lines crossing the windows, which has not been allowed for.]

Adding an equal amount for the other uncovered side of the coil, the flux across dz due to C_1 is

$$\frac{16B}{b_1}n_1C_1\frac{zdz}{z+\frac{2D}{\pi}},$$

and it is looped on

$$\frac{n_1C_1}{b_1}2z \text{ of } n_1C_1,$$

so that the energy associated with it is

$$\frac{16B}{b_1^2}n_1^2C_1^2\frac{z^2dz}{z+\frac{2D}{\pi}}$$

Integrating this between the limits $z=b_1/2$ and $z=0$ we get

$$4Bn_1^2C_1^2\left\{\tfrac{1}{2}-2\lambda_1+4\lambda_1^2\log\frac{1+2\lambda_1}{2\lambda_1}\right\}, \qquad \text{where } \lambda_1=\frac{2D}{\pi b_1}$$

which is the energy of C_1 due to its lines through the primary coil.

Again the energy of C_1 due to the lines it sends through the space occupied by the secondary coil is

$$=4Bn_1^2C_1^2\int_{\frac{b_1}{2}}^{\frac{b_1}{2}+b_2}\frac{dy}{y+\frac{2D}{\pi}}$$

$$=4Bn_1^2C_1^2\log\frac{1+2\lambda_1+2\nu_1}{1+2\lambda_1}$$

where $\nu_1=\frac{b_2}{b_1}$.

The sum of these two results being $\frac{1}{2}L''_1C_1^2$

$$L''_1=8Bn_1^2\left\{\tfrac{1}{2}-2\lambda_1+4\lambda_1^2\log\frac{1+2\lambda_1}{2\lambda_1}+\log\frac{1+2\lambda_1+2\nu_1}{1+2\lambda_1}\right\}.$$

Similarly, if $\lambda_2=\frac{2D}{\pi b_2}$, $\nu_2=\frac{b_1}{b_2}=\frac{1}{\nu_1}$

$$L''_2 = 8Bn_2^2\left\{\tfrac{1}{2} - 2\lambda_2 + 4\lambda_2^2 \log\frac{1+2\lambda_2}{2\lambda_2} + \log\frac{1+2\lambda_2+2\nu_2}{1+2\lambda_2}\right\}$$

In a like manner we find that

$$M''_{12} = 8Bn_1n_2\left\{1 - (\lambda_2 + \tfrac{1}{2}\nu_2)\log\frac{1+2\lambda_1+2\nu_1}{1+2\lambda_1}\right\}$$

$$M''_{21} = 8Bn_1n_2\left\{1 - (\lambda_1 + \tfrac{1}{2}\nu_1)\log\frac{1+2\lambda_2+2\nu_2}{1+2\lambda_2}\right\}$$

and substituting in the equations for x_1 and x_2 in § 43

$$x''_1 = \frac{8B}{\sigma}\left\{-\tfrac{1}{2} - 2\lambda_1 + 4\lambda_1^2\log\frac{1+2\lambda_1}{2\lambda_1} + \log\frac{1+2\lambda_1+2\nu_1}{1+2\lambda_1} + (\lambda_1 + \tfrac{1}{2}\nu_1)\log\frac{1+2\lambda_2+2\nu_2}{1+2\lambda_2}\right\}$$

$$x''_2 = \frac{8B}{\sigma}\left\{\text{Interchange } \lambda_1 \text{ and } \lambda_2,\ \nu_1 \text{ and } \nu_2 \text{ in above.}\right\}$$

and finally for the leakage coefficients of the transformer

$$x_1 = x'_1 + x''_1$$
$$x_2 = x'_2 + x''_2.$$

46. From the expressions in §§ 44, 45, let us determine x_1 and x_2 for transformers with square windows and iron tongue of square cross section, and of which the secondary coil occupies three-fourths as much space as is occupied by the primary.

Then $D = b_1 + b_2$, $b_1 = \frac{4}{7}D$, $b_2 = \frac{3}{7}D$,

$$\beta = \beta', \quad \nu_1 = 3/4, \quad \nu_2 = 4/3,$$
$$\lambda_1 = \frac{2D}{\pi b_1} = 1.11, \quad \lambda_2 = \frac{2D}{\pi b_2} = 1.48$$

From § 44 we get

$$x'_1 = 9.6\frac{\beta}{\sigma}, \quad x'_2 = 19.8\frac{\beta}{\sigma},$$

and from § 45

$$x''_1 = 2\frac{B}{\sigma}, \quad x''_2 = \frac{7}{2}\frac{B}{\sigma}.$$

Remembering that $B = 2\beta + \frac{\pi}{2}D$, we find that

$$x_1 = x'_1 + x''_1 = \frac{13.6\beta + 3.1D}{\sigma},$$

$$x_2 = x'_2 + x''_2 = \frac{26.8\beta + 5.5D}{\sigma}.$$

As $\sigma = 4\pi\mu \cdot \frac{4\beta^2}{4(\beta + D)}$, and as, in a transformer shaped as we have supposed 2β will be very nearly 1.2 D,

$$x_1=\frac{4}{\mu}, \quad x_2=\frac{7.6}{\mu}.$$

If we take $\mu=2250$,

$$x_1=.00178, \quad x_2=.00338$$

$$x_1+x_2=.005$$

which in a 10 K.W. 50 period transformer, $\tau=6000$, would give a drop on non-inductive load of 4.3 per cent., and on an inductive load of .8 power factor a drop of 18 per cent. approximately.

[Take $\tau_1=\tau_2=\tau$, $y=\sqrt{\frac{\tau \text{Sin}\delta}{2}}$, Sin$\delta$=.75, in formula in § 23 for a rough estimate of the drop].

47. In order to determine x_1 and x_2 for a shell transformer with interleaved windings we have, as before, to determine x'_1 and x'_2 due to the lines that cross the windows, and x''_1, x''_2, due to the remaining leakage lines, separately. We will first determine x'_1 and x'_2 in the general case. Let the sections of the two coils be arranged, as in Fig. 7, symmetrically, with regard to the median plane across the window so that a section of one of the coils occupies the central position in the window. Let the coil to which the central section belongs be called the even coil and its sections the even sections, and let the other coil be called the odd coil and its sections the odd sections. The even coil will have an odd number, i say, of sections, and the odd coil will have an even number, j say, of sections where j may be equal to either $i-1$ as in Fig. 7, or to $i+1$, in which case the two outer sections would be odd ones. Let b_2 be the thickness of each of the even sections and b_1 that of each of the odd ones ; and let L_2, x_2, n_2, C_2 refer to the even coil and L_1, x_1, n_1, C_1, etc., to the odd one. Let 2β, $2\beta'$, be the section of the iron tongue, $2\beta'$ being measured perpendicular to the laminae ; and let D be the breadth of the window.

In Fig. 7 is a diagram of the window in which the sections are numbered in accordance with the above plan, the central section being numbered O, and directly below is what we may call the M.M.F. diagram for C_2 in which the zig-zag line

OP gives the distribution of M.M.F. in the different sections due to C_2.

Thus if the M.M.F. round the Central section O, that is $\frac{4\pi n_2 C_2}{i}$, be called m_2,

the M.M.F. round the magnetic circuit 1, 1′, is m_2,
the M.M.F. round the magnetic circuit 3, 3′ is $3m_2$,
and so on, so that the ordinate of OP opposite any odd section is proportional to the number of the section. And it is easily seen that the mean ordinate of OP opposite any even section is also proportional to the number specifying that section.

The flux per unit length across the window at any point in OQ is

$$= \text{M.M.F.} \frac{2\beta'}{2\text{D}}$$

and if $m_2 \frac{\beta'}{\text{D}}$ *i.e.* $\frac{4\pi n_2 C_2}{i} \frac{\beta'}{\text{D}} = f_2$

it is easy to see that, due to C_2,

the total flux round the circuit 1, 1′, is $f_2 b_1$
" " " " " " 2, 2′, is $2f_2 b_2$
" " " " " " 3, 3′, is $3f_2 b_1$
and so on.

(*a*) The energy of C_2 due to the leakage lines it sends through the even (its own) sections is, as we neglect the reluctance of the iron, the same as if the odd sections were removed and the even ones pushed up together.

By § 44, (I.), this energy is

$$= \frac{\pi}{6} \frac{2\beta' i b_2}{\text{D}} n_2^2 C_2^2. \qquad \text{(I.)}$$

The energy of C_2, due to the leakage lines it sends through the odd sections, is, by aid of Fig. 7, found to be

$$= \tfrac{1}{2} f_2 b_1 \frac{n_2 C_2}{i} \left\{ 1^2 + 3^2 + 5^2 + \quad + \overline{j-1}^2 \right\}.$$

$$= \frac{2\pi\beta'}{\text{D}} \frac{n_2 C_2^2}{i^2} b_1 \frac{j(j^2-1)}{6} \qquad \text{(II.)}$$

This follows easily when it is noticed, Fig. 7, that the flux $3f_2 b_1$ (for instance) round 3, 3′ is looped on three even sections, *i.e.*, on the current $3n_2 C_2/i$, and similarly for the others.

Adding I. and II. we get the energy of C_2, due to the leakage lines it sends across one window. The other window contributes an equal amount, hence,

$$\tfrac{1}{2} L'_2 C_2^2 = \frac{2\pi\beta'}{3D} n_2^2 C_2^2 \left\{ ib_2 + \frac{j(j^2-1)}{i^2} b_1 \right\},$$

or

$$L'_2 = \frac{4\pi\beta'}{3D} n_2^2 \left\{ ib_2 + \frac{j(j^2-1)}{i^2} b_1 \right\}. \qquad \text{(III.)}$$

(*b*) The mutual energy of C_2 and C_1, due to the C_2 lines through the even spaces being looped on C_1, can be read off from Fig. 7, and is

$$= f_2 b_2 \frac{n_1 C_1}{j} \left\{ 2^2 + 4^2 + 6^2 + \quad + \overline{i-1}^2 \right\}$$

$$= \frac{2\pi\beta'}{3D} n_1 n_2 C_1 C_2 \frac{i^2-1}{j} b_2. \qquad \text{(IV.)}$$

The mutual energy of C_2 and C_1, due to the C_2 lines through the odd spaces being looped on C_1 is

$$= f_2 b_2 \frac{n_1 C_1}{j} \left\{ 1^2 + 3^2 + 5^2 + \quad + \overline{j-1}^2 \right\}$$

$$= \frac{2\pi\beta'}{3D} n_1 n_2 C_1 C_2 \frac{j^2-1}{i} b_1. \qquad \text{(V.)}$$

[This will be seen by considering circuit 3, 3′. The flux round it is $3f_2 b_1$, and it completely encircles the two odd C_1 sections 1 and 1′, and partially the C_1 windings in 3 and 3′. But in § 44, III., it was shown that the mutual energy due to a uniform flux encircling a uniformly distributed current occupying the same space, was the same as if the whole flux encircled half the current; hence, in this case, the energy contributed by 3 and 3′ is the same as if the flux $3f_2 b_1$ encircled one only of them completely, so that altogether the flux $3f_2 b_1$ encircles $3n_1 C_1/j$ of the current C_1.]

Adding IV. and V., and doubling the sum to allow for the other window,

$$M'_{12} C_1 C_2 = \frac{4\pi\beta'}{3D} n_1 n_2 C_1 C_2 \left\{ \frac{i^2-1}{j} b_2 + \frac{j^2-1}{i} b_1 \right\},$$

or
$$M'_{12} = \frac{4\pi\beta'}{3D} n_1 n_2 \left\{ \frac{i^2-1}{j} b_2 + \frac{j^2-1}{i} b_2 \right\} \qquad \text{(VI.)}$$

(*c*) In a similar manner, by aid of the M.M.F. diagram for C_1 also given in Fig. 7, or from symmetry we find that

$$\left.\begin{aligned} L'_1 &= \frac{4\pi\beta'}{3D} n_1^2 \left\{ jb_1 + \frac{i(i^2-1)}{j^2} b_2 \right\} \\ M'_{21} &= \frac{4\pi\beta'}{3D} n_1 n_2 \left\{ \frac{j^2-1}{i} b_1 + \frac{i^2-1}{j} b_2 \right\} \\ &= M'_{12} \end{aligned}\right\} \text{VII.}$$

(*d*) Substituting the values for L'_2, L'_1, M'_{12}, M'_{21}, in V., VI., and VII. in the expressions for x'_2 and x'_1 in § 43 we find

I. If the two extreme sections in the window are even ones, in which case $j = i - 1$, that

$$x'_2 = -\frac{4\pi\beta'}{3\sigma D} \left\{ b_2 + \frac{i-2}{i} b_1 \right\},$$

$$x'_1 = \frac{4\pi\beta'}{3\sigma D} \left\{ \frac{i+1}{i-1} b_2 + b_1 \right\},$$

$$x'_2 + x'_1 = \frac{8\pi\beta'}{3\sigma D} \frac{H}{i(i-1)},$$

where $H = ib_2 + (i-1)b_1$

= total height of window.

II. If the two extreme sections in the window are odd ones, in which case $j = i + 1$,

$$x'_2 = \frac{4\pi\beta'}{3\sigma D} \left\{ b_2 + \frac{i+2}{i} b_1 \right\},$$

$$x'_1 = -\frac{4\pi\beta'}{3\sigma D} \left\{ \frac{i-1}{i+1} b_2 + b_1 \right\}$$

$$x'_2 + x'_1 = \frac{8\pi\beta'}{3\sigma D} \frac{H}{i(i+1)}$$

(*e*). From the results in (*d*) we see

I. That the coefficient x'_2 or x'_1 of the coil to which the two extreme sections belong is negative, indicating that this part of the leakage produces on the coil, with the extreme sections, a capacity and not an inductive effect.

The same thing is true with regard to the coefficients x''_2 and x''_1 due to the leakage lines that do not cross the windows and which will be determined in the next paragraph, and it has been pointed out in § 29 that secondary leakage reduces the efficiency of the transformer when operating inductive loads, hence the winding of a shell transformer for inductive work should be so arranged that the two extreme sections belong to the secondary or output coil.

4

II. That the sum $x'_1+x'_2$ of the coefficients due to the leakage lines crossing the windows is inversely proportional to the product of the numbers of sections in the two coils.

(*f*). If $2b=D$ be the breadth, and $2b'$ the height of the window, 2β, $2\beta'$ the section of the iron tongue as before, q_1 and q_2 the space factors of the coils; and μ the permeability of the iron; the formulae in (*d*) can be put in the following forms suitable for calculation.

$$x'_2=\frac{b'(b+b'+\beta)}{3\mu b\beta}\,\frac{\dfrac{q_1+2q_2}{q_1+q_2}\mp i}{i(i\mp 1)}$$

$$x'_1=\frac{b'(b+b'+\beta)}{3\mu b\beta}\,\frac{\dfrac{q_1}{q_1+q_2}\pm i}{i(i\mp 1)}$$

$$x'_2+x'_1=\frac{b'(b+b'+\beta)}{3\mu b\beta}\,\frac{2}{i(i\mp 1)}$$

where the upper signs are to be taken when the two extreme sections belong to coil 2 (the even coil), of i sections, to which the middle section belongs, and the lower signs, when the extreme sections belong to coil 1.

Evidently $i\mp 1$ is the number of sections of coil 1.

48. When we make the same assumption as is made in §45 with regard to the paths taken by the leakage lines that do not cross the windows, we can obtain the values of the coefficients x''_1, x''_2 due to these lines in the general case of interleaved windings.

Specifying the coils, sections, and dimensions of the transformer exactly as in the preceding paragraph, and letting

$$B=2\beta+\frac{\pi}{2}D=2\beta+\pi b \text{ (see § 47 } f\text{).}$$

$$\lambda_2=\frac{2D}{\pi b_2},\quad \lambda_1=\frac{2D}{\pi b_1},$$

$$\nu_2=\frac{b_1}{b_2},\quad \nu_1=\frac{b_2}{b_1},$$

we find, after proceeding as in §§ 45, 47, that

$$x''_2=\frac{32B}{\sigma}\Big[-\frac{1}{8}-\frac{1}{8}\nu_2\big(1-\frac{1}{i^2}\big)-\frac{\lambda_2}{2i}+\frac{\lambda_2^2}{i^2}\log\frac{1+2\lambda_2}{2\lambda_2}+$$

$$\sum_{n=2}^{n=i-1}\left\{\left(\frac{\frac{1}{2}n\nu_2+\lambda_2}{i}\right)^2+\frac{n(\frac{1}{2}n\nu_2+\lambda_2)}{2ij}\right\}\log\frac{2\lambda_2+n(1+\nu_2)+1}{2\lambda_2+n(1+\nu_2)-1}$$

$$+\sum_{m=1}^{m=j-1}\left\{\frac{m^2}{4i^2}+\frac{m(\frac{1}{2}m\nu_1+\lambda_1)}{2ij}\right\}\log\frac{2\lambda_1+m(1+\nu_1)+1}{2\lambda_1+m(1+\nu_1)-1}\Bigg]$$

$$x''_1=\frac{32\mathrm{B}}{\sigma}\Bigg[-\frac{1}{8}-\frac{\nu_1}{8}-\frac{\lambda_1}{2j}+$$

$$\sum_{n=2}^{n=i-1}\left\{\frac{n(\frac{1}{2}n\nu_2+\lambda_2)}{2ij}+\frac{n^2}{4j^2}\right\}\log\frac{2\lambda_2+n(1+\nu_2)+1}{2\lambda_2+n(1+\nu_2)-1}$$

$$+\sum_{m=1}^{m=j-1}\left\{\frac{m(\frac{1}{2}m\nu_1+\lambda_1)}{2ij}+\left(\frac{\frac{1}{2}m\nu_1+\lambda_1}{j}\right)^2\right\}\log\frac{2\lambda_1+m(1+\nu_1)+1}{2\lambda_1+m(1+\nu_1)-1}\Bigg].$$

where, in each of the above expressions, the values to be given to n are all the even numbers from 2 to $i-1$, and to m all the odd numbers from 1 to $j-1$ inclusive.

It will be easily seen, by considering the case of a winding with three sections, that by spreading out the free ends of the sections as is done for cooling purposes the values of the coefficients x''_1 and x''_2 will be slightly increased.

49. If the results in § 48 be written

$$x''_2=\frac{\mathrm{B}}{\sigma}\mathrm{X}_2=\frac{(2\beta+\pi b)\ (b+b'+\beta)}{4\pi\mu\beta\beta'}\mathrm{X}_2$$

$$x''_1=\frac{\mathrm{B}}{\sigma}\mathrm{X}_1=\frac{(2\beta+\pi b)\ (b+b'+\beta)}{4\pi\mu\beta\beta'}\mathrm{X}_1$$

it is easily seen that X_1 and X_2 depend only on the shape of the window, the numbers of sections of the two coils, and the ratio of their space-factors. In the following table are given the values of X_2 and X_1 for some different values of i and j, for square windows and for oblong ones whose height, measured parallel to the iron tongue, is twice their breadth ($2b'=4b$), and for some different ratios of space-factors.

It will be noticed in the following table that the coefficient of the coil to which the extreme sections belong is negative, and that the sum of X_1 and X_2 is (q.p.) inversely proportional to the product of the numbers of sections: also that the change of the space-factor ratio from 4/3 to 3/4 does not cause much change in X_1 and X_2, unless in the three-section winding. Hence, by aid of this table we can obtain very approximate values of

X_1 and X_2 for other windings, and for windows of other shapes by interpolation.

Numbers of Sections.		Ratios of space-factors q_2/q_1	Square Window $b=b'$			Oblong Window $b'=2b$		
Coil 2 i	Coil 1 $i\mp1$		X_1	X_2	X_1+X_2	X_1	X_2	X_1+X_2
1	2	3/4	—.33	1.75	1.42	—.41	2.75	2.34
1	2	4/3	—.37	1.92	1.55	—.58	3.03	2.45
3	2	4/3	.76	—.29	.47	1.13	—.41	.72
3	2	3/4	.80	—.32	.48	1.21	—.48	.73
3	4	3/4	—.27	.50	.23	—.38	.75	.37
3	4	4/3	—.29	.53	.24	—.41	.79	.38
5	4	4/3	.36	—.21	.15	.54	—.32	.22
5	4	4/5	.38	—.23	.15	.56	—.35	.21
5	6	6/5	—.19	.27	.08	—.29	.44	.15

[Note that the central section always belongs to coil 2 of i sections, i odd.]

50. There is also magnetic leakage due to the lines in the copper conductors, and in the spaces between them, which is of considerable importance in the case of large, low-pressure transformers.

It is well known that the inductance per unit length of a wire, due to the lines in itself is 1/2, and due to the lines between its surface of radius r and a concentric cylinder of radius r' is $2\log r'/r$. In the case of insulated wire wound in a coil it will be very near the truth to take for r' the radius of the circle equal in area to the total area allowed each wire in the winding, so that if q be the space-factor

$$\frac{\pi r^2}{\pi r'^2}=q,$$

and the inductance of the wire per unit length will be

$$\tfrac{1}{2}+\log\frac{1}{q}.$$

Hence, if l be the mean length of a turn of either coil, their inductances arising from this cause are

$$L'''_1=n_1 l\left\{\tfrac{1}{2}+\log\frac{1}{q_1}\right\},$$

$$L'''_2 = n_2 l \left\{ \tfrac{1}{2} + \log \frac{1}{q_2} \right\},$$

and if x'''_1, x'''_2 be the corresponding leakage coefficients

$$x'''_1 = \frac{L'''_1}{n_1^2 \sigma} = \frac{l}{n_1 \sigma} \left\{ \tfrac{1}{2} + \log \frac{1}{q_1} \right\},$$

$$x'''_2 = \frac{L'''_2}{n_2^2 \sigma} = \frac{l}{n_2 \sigma} \left\{ \tfrac{1}{2} + \log \frac{1}{q_2} \right\}.$$

Now it is easy to show by means of relations already given, that

$$\frac{l}{n_1 \sigma} = \frac{2 w P_2}{\rho c \tau E_1} \text{ and } \frac{l}{n_2 \sigma} = \frac{2 w P_2}{\rho c \tau E_2}.$$

where P_2=full load (non-inductive) output, c=amp. of current density, ρ=sp. res. of copper ; so that we have

$$x'''_1 = \frac{w P_2}{\rho c \tau E_1} \left\{ 1 + 2 \log \frac{1}{q_1} \right\},$$

$$x'''_2 = \frac{w P_2}{\rho c \tau E_2} \left\{ 1 + 2 \log \frac{1}{q_2} \right\}.$$

It will be shown, for similar transformers designed on the same lines, that τ is proportional to $\sqrt{w P_2}$, hence if the e.m.f.s remain fixed, x'''_1 and x'''_2 will increase as the square root of the product of output and frequency increases.

If the conductors be rectangular in section instead of circular, the above expressions for x_1''' and x_2''' will be sufficiently accurate for all practical purposes.

Note.—The connectors from the ends of either coil to the corresponding terminals outside the cases of large transformers ought to include as small an area as possible, since on account of the proximity of the iron of the transformer and of the case, the loops so formed would have considerable inductance, thus increasing the leakage coefficients, especially that of the low pressure coil, and so impairing the regulation on inductive loads.

51. Let us determine the leakage coefficients x_p, x_s, for the transformer designed in Section II. [In this paragraph x_p, x_s, will be the primary and secondary coefficients respectively, while x_2, q_2, etc., will still refer to the coil with the middle section].

The details for this transformer are (see §§ 28 et seq.),

$i=1$, $b=b'$, $\beta=\beta'$, $\beta=1.151b$, $\mu=2250$,

$\rho=1800$, $c=12.9$, $E_1=3111.10^8$, $E_2=311.10^8$.

$P_2=12.5$.K.W., $\tau=6080$, $w=100\pi$, and as the middle coil is the primary one $q_2=q_p=.5$, $q_1=q_s=.7$.

From § 47, *f*, taking the lower signs,

$$x'_1=\frac{2b+\beta}{3\mu\beta}\;\frac{\frac{17}{12}+1}{2}=.000490$$

$$x'_1=\frac{2b+\beta}{3\mu\beta}\;\frac{\frac{7}{12}-1}{2}=-.000084.$$

From § 49, taking from the table the values of X_2 and X_1, $i=1$, $b=b_1^i$, and $q_2/q_1=3/4$, which is sufficiently close to 5/7, we get

$$x''_2=\frac{(2\beta+\pi b)(2b+\beta)}{4\pi\mu\beta^2}1.75=.00080$$

$$x''_1=-\frac{(2\beta+\pi b)(2b+\beta)}{4\pi\mu\beta^2}.33=-.00015$$

From § 50,

$$x'''_2=x'''_p=\frac{9}{10^7}(1+2\log 2)=.000002,$$

$$x'''_1=x'''_s=\frac{9}{10^6}\left(1+2\log\frac{10}{7}\right)=.000015,$$

which in this case of few sections are relatively negligible.

But $x_p=x'_2+x''_2+x'''_2$, $x_s=x'_1+x''_1+x'''_1$,

hence $x_p=.00129$, $x_s=-.00024$, $x_p+x_s=.00105$.

If this transformer had been wound so that the secondary as a single coil occupied the central position, with half of the primary on either side, its leakage coefficients x'_p, x'_s would be

$x'_p=-.00029$, $x'_s=.0014$.

and it is interesting to find what effect this change in the relative positions of the two coils would have on the efficiency for inductive loads.

When we neglect all small terms but the one that depends on the first power of the leakage, we find from §15 that the maximum efficiency

$$\eta=\frac{1}{1+\frac{2}{\text{Cos}\phi}\sqrt{\text{TSin}\delta}+2x_s\tan\phi\text{Sin}\delta}$$

$$=\left\{1-\frac{2}{\text{Cos}\phi}\sqrt{\text{TSin}\delta}\right\}(1-2x_s\tan\phi\text{Sin}\delta)$$

$$=\eta_0(1-2x_s\tan\phi\text{Sin}\delta)\qquad\text{(q.p.)}$$

where $T=|\frac{1}{\tau_1}+\frac{1}{\tau_2}=\frac{2}{\tau}$

Hence η is proportional to $1-2x_s\tan\phi\text{Sin}\delta$ and the ratio η/η' of the efficiencies of the same transformer, but differently wound as above, is for loads of the same power factor,

$$\frac{\eta}{\eta'}=\frac{1-2x_s\tan\phi\text{Sin}\delta}{1-2x'_s\tan\phi\text{Sin}\delta}$$

$$=1-2(x_s-x'_s)\tan\phi\text{Sin}\delta,$$

$$=1+.00328\tan\phi\text{Sin}\delta,$$

as $x_s=-.00024$, $x'_s=.0014$.

If Cosϕ, the power factor of the load, be .8, tanϕ=.75,

and $\frac{\eta}{\eta'}=1.0019$ (taking $\delta=50°$),

and if Cosϕ=.6, tanϕ=4/3,

and

$$\frac{\eta}{\eta'}=1.0034.$$

So that when the secondary coil occupies the two outside positions the transformer will have for inductive loads, when Cosϕ=.8, a greater efficiency by .19 per cent., and when Cosϕ=.6 a greater efficiency by .34 per cent. than when the primary occupies the outside positions.

For non-inductive loads the difference in efficiency will be very small, as it then depends on the square of x_s.

52. In § 41 it has been shown how, from the data for any particular design, the dimensions of the carcass and approximate values of the numeric τ and of the efficiency can be quickly obtained.

Selecting from the series in § 41, transformer (c), of which the the details are :—

Capacity 12.5 K.W. at 50 periods,

$2b'=3b$, $2\beta'=3\beta$, $\beta=1.141b$, $\tau=6910$,

$\mu=2250$, $c=12.9$, Sinδ=.766, $q_p=.5$, $q_s=.7$,

to which we will add $E_p=2200$ volts$=3111 \,.\, 10^8$, $E_s=311 \,.\, 10^8$. let us determine its leakage coefficients and approximate values of its voltage drop for different kinds of loads if it be wound in five sections, three secondary and two primary.

As the middle and end sections belong to the secondary coil, $q_2=.7$, $q_1=.5$ and from the formulae in § 47, *f*., taking the upper sign we get

$$x'_2=-\frac{.377}{\mu}=-.000168,$$

$$x'_1=\frac{.909}{\mu}=\ .000404.$$

From the formulae in § 49,

$$x''_2=\frac{.804}{\mu}X_2,\quad x''_1=\frac{.804}{\mu}X_1;$$

for X_2 and X_1, which are to be for a window in which $b'/b=1.5$, we will take the mean of the values given in the table for windows in which $b'/b=1$ and $b'/b=2$, and as $i=3$, five sections, $q_2/q_1=4/3$, we get

$$X_2=-\frac{.29+.41}{2}=-.35$$

$$X_1=\frac{.76+1.13}{2}=\ .945$$

hence

$$x''_2=-.000125,\quad x''_1=.000338.$$

From the formulae in § 50,

$$x'''_p=.000002,\quad x'''_s=.000014.$$

Hence, as

$$x_p=x'_1+x''_1+x'''_p,\quad x_s=x'_2+x_2''+x'''_s$$

$$x_p=.000744,\qquad x_s=-.000279,$$

$$x_p+x_s=.000465.$$

In Section I., § 23, it was shown that R, the drop per cent., can be expressed in the form

$$R=100\left\{x\mathrm{Sin}\phi+t\mathrm{Cos}\phi+x^2\left(\mathrm{Sin}^2\phi+\frac{\mathrm{Cos}^2\phi}{2}\right)+t^2\left(\mathrm{Cos}^2\phi+\frac{\mathrm{Sin}^2\phi}{2}\right)+xt\mathrm{Sin}\phi\mathrm{Cos}\phi\right\}$$

where $x=y(x_p+x_s)$, $t=y\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)=2\frac{y}{\tau}$.

Now to the first order $y=\theta$ (full load values)

and $\theta=\sqrt{\frac{z\tau\mathrm{Sin}\delta}{2}}$ q.p. (see § 17)

where z (=1 in this case) is the chosen ratio of copper to iron losses at full load.

Hence in this case

$$y=\sqrt{\frac{6910 \cdot .766}{2}}=51 \qquad \text{(q.p.)}$$

and $x=51 \cdot .000465=.0237$,

$$t=51 \cdot \frac{2}{6910}=.0148.$$

which, by means of the formula for R give the following approximate values for the regulation.

Power Factor. Cosϕ		Drop per cent. R.
1.0	...	1.53
.9	...	2.43
.8	...	2.68
.6	...	2.86

These figures agree remarkably well with the following (already discussed in Section I.), given by the Westinghouse Co. as the regulation of their 10 K.W. 60 period O.D. transformers, which are also wound in five sections—three primary and two secondary. As regards regulation, the 12.5 K.W. 50 period transformer considered above, and the 10 K.W. 60 period Westinghouse one are nearly equivalent, as wP is nearly the same for both [$2\pi \cdot 625$ as against $2\pi \cdot 600$].

REGULATION OF EQUIVALENT WESTINGHOUSE TRANSFORMER.

Power Factor Cosϕ		Drop per cent. —
1.0	...	1.65
.9	...	2.45
.8	...	2.65
.6	...	2.80

53. In order to obtain approximate values for the leakage coefficients of transformers of the core type it will be sufficiently accurate to consider the core as straight, and connecting two large masses of iron.

Let the core be circular in section of radius r having coil 2 lying betwen the cylinders whose radii are r and $r+b_2$ and coil 1, lying between the cylinders whose radii are $r+b_2$ and $r+b_2+b_1$, and let $r+b_2+b_1=r_0$.

The flux density due to C_1 at all points on the cylinder of radius r_0-z is

$$=\frac{4\pi n_1 C_1}{\lambda'}\frac{z}{b_1}$$

where λ' is the length of the windings parallel to the core, and the flux in the C_1 space between the cylinders of radii r_0-z and $r_0-(z+dz)$ is

$$=\frac{4\pi n_1 C_1}{\lambda'}\frac{z}{b_1}2\pi(r_0-z)dz,$$

and it is looped on that part

$$\frac{n_1 C_1}{b_1}z$$

that lies without it, hence it contributes energy dE where

$$dE=\frac{4\pi^2 n_1^2 C_1^2}{\lambda' b_1^2}(r_0-z)\, z^2 dz.$$

Integrating between $z=O$ and $z=b_1$

$$E=\frac{4\pi^2 n_1^2 C_1^2}{\lambda'}\left\{\frac{r_0 b_1}{3}-\frac{b_1^2}{4}\right\}$$

$$=\frac{4\pi^2 n_1^2 C_1^2}{\lambda'}\left\{\frac{(r+b_2)b_1}{3}+\frac{b_1^2}{12}\right\},$$

as $r_0=r+o_1+b_2$.

The current C_1 also sends through the space occupied by coil 2 a uniform flux

$$=\frac{4\pi n_1 C_1}{\lambda'}\pi\left\{(r+b_2)^2-r^2\right\}$$

which is looped on all of $n_1 C_1$ and therefore contributes energy to the amount

$$\frac{4\pi^2 n_1^2 C_1^2}{\lambda'}\left\{r\,b_2+\frac{b_2^2}{2}\right\},$$

so that the energy of C_1 due to those lines that it produces and that do not traverse the core is

$$=\frac{4\pi^2 n_1^2 C_1^2}{\lambda'}\left\{r\left(\frac{b_1}{3}+b_2\right)+\frac{b_1^2}{12}+\frac{b_1 b_2}{3}+\frac{b_2^2}{2}\right\},$$

but it is also $=\frac{1}{2}L_1 C_1^2$

hence

$$L_1=\frac{8\pi^2 n_1^2}{\lambda'}\left\{r\left(\frac{b_1}{3}+b_2\right)+\frac{b_1^2}{12}+\frac{b_1 b_2}{3}+\frac{b_2^2}{2}\right\}.$$

In a similar manner we find that

$$L_2 = \frac{8\pi^2 n_2^2}{\lambda'}\left\{\frac{rb_2}{3} + \frac{b_2^2}{12}\right\}$$

$$M_{12} = M_{21} = \frac{8\pi^2 n_1 n_2}{\lambda'}\left\{\frac{rb_2}{2} + \frac{b_2^2}{6}\right\},$$

but

$$x_1 = \frac{1}{n_1\sigma}\left\{\frac{L_1}{n_1} - \frac{M_{21}}{n_2}\right\},$$

$$\therefore x_1 = \frac{8\pi^2}{\lambda'\sigma}\left\{\left(\frac{b_1}{3} + \frac{b_2}{2}\right)r + \frac{(b_1 + 2b_2)^2}{12}\right\}, \text{ and similarly}$$

$$x_2 = -\frac{8\pi^2}{\lambda'\sigma}\left\{\frac{rb_2}{6} + \frac{b_2^2}{12}\right\}$$

$$x_1 + x_1 = \frac{8\pi^2}{\lambda'\sigma}(b_1 + b_2)\left\{\frac{r}{3} + \frac{b_1 + 3b_2}{12}\right\}.$$

From which we see that the leakage coefficient, x_2, of the coil next the core is negative.

The equation giving $x_2 + x_1$ can be written

$$x_1 + x_2 = \frac{4\pi^2}{3\lambda'\sigma}(b_1 + b_2)\ (S_1 + S_2) \qquad \text{(I.)}$$

where S_1 and S_2 are the mean radii of the two coils ; and if q_1 and q_2 be their space factors, their copper sections per unit length of winding are

$b_1 q_1$ and $b_2 q_2$.

These will be equal or very nearly so, and

let $\quad b_1 q_1 = b_2 q_2 = s,$

then $\quad b_1 + b_2 = 2s/Q$

where $Q = \dfrac{2q_1 q_2}{q_1 + q_2}$, the harmonic mean of q_1 and q_2.

The total copper volume

$$= 2\pi\lambda'(S_1 q_1 b_1 + S_2 q_2 b_2) = 2\pi\lambda' s(S_1 + S_2) = \pi Q\lambda'(b_1 + b_2)(S_1 + S_2),$$

and $\sigma = 4\pi\mu \dfrac{\text{Volume of iron}}{\lambda^2}$

where λ = length of magnetic circuit,
hence

$$x_1 + x_2 = \frac{\lambda^2}{3\mu Q\lambda'^2} \times \frac{\text{volume of copper}}{\text{volume of iron}} \qquad \text{(II.)}$$

and if the transformer is being designed so that the copper losses are to be z times the iron losses at full load, then

$$\frac{\text{Vol. copper}}{\text{Vol. iron}} = z\frac{\text{I}}{\text{K}}$$

where K and I are these losses per cm.^{3}

$$\therefore \quad x_1 + x_2 = \frac{\lambda^2}{3\mu Q\lambda'^2} \quad z\frac{\text{I}}{\text{K}} \qquad \text{(III.)}$$

a form very suitable for the determination of x_1+x_2 for core transformers.

For example, if $z=1$, $\text{I}=10^5$, $\text{K}=15.10^4$, $q_1=.5$, $q_2=.7$, $Q=.583$ as before, x_1+x_2 for any core transformer designed on these data is given by

$$x_1+x_2=\frac{\lambda^2}{3\mu\lambda'^2}1.14.$$

For a core transformer of the H type, simply wound, in which the rectangular opening in the stampings is 10.2×25.2 cm., and the width of the surrounding iron strip 8.9 cm. (See § 57)

$\lambda'=2\times 25.2=50.4$ cm.

$\lambda=106.4$ cm.

and if $\mu=2250$,

$x_1+x_2=.00075.$

54. If a core transformer be wound with $2i$ layers, i each of primary and secondary arranged alternately, and if D be the total depth of the windings, it can be shown that

$$x_1+x_2=\frac{4\pi^2 D}{3\lambda'\sigma i^3} \times \text{ sum of the mean radii of all the layers,}$$

$$=\frac{8\pi^2 D}{3\lambda'\sigma i^2} \times \text{ mean of the mean radii of all the layers,}$$

which by exactly similar reasoning to that in § 53 can be put into either of the forms,

$$x_1+x_2=\frac{1}{i^2}\,\frac{\lambda^2}{3\mu Q\lambda'^2}\,\frac{\text{volume of copper}}{\text{volume of iron}}.$$

$$\text{or } x_1+x_2=\frac{1}{i^2}\,\frac{\lambda^2}{3\mu\lambda'^2}\,\frac{z\text{I}}{\text{QK}}.$$

If there be i layers of one coil and $i+1$ of the other, then we may take

$$x_1+x_2=\frac{1}{i(i+1)}\,\frac{\lambda^2}{3\mu\lambda'^2}\,\frac{z\text{I}}{\text{QK}}.$$

This result and those in § 53 will be sufficiently accurate for all practical purposes when the coils are rectangular in plan.

The Transformer Numerics.

55. The numeric $\tau\ (=\tau_1=\tau_2\ (\text{q.p.})\ )$ for transformers of any given type can be expressed in terms of the full-load output, periodicity, and the magnetic and electric qualities of the iron and copper.

Let us consider the case of transformers of the shell type similar to the one designed in Section II., with square windows $(2b, 2b)$, and iron tongue of square cross section $(2\beta, 2\beta)$.

From § 33

$$\tau = A\frac{\mu w}{\rho}\ \frac{b^2\beta^2}{(b+\beta)\ (2b+\beta)} \qquad \text{(I.)}$$

where A is a constant depending on the iron and copper space factors.

From the solution, as in § 32 of the equation

$$\frac{2QKb^2\ (b+\beta)}{p I\beta^2(2b+\beta)} = z$$

which expresses the relation between the iron and copper losses at full load, we get

$$\beta = Bb \qquad \text{(II.)}$$

in which B will be a constant, if, for all transformers of the series

$$\frac{pIz}{QK}$$

be constant.

We may consider p the iron space factor as fixed, and, provided the primary and secondary pressures remain the same, Q, the harmonic mean of the copper space factors, also as fixed ; and the above expression will be constant if z, the ratio of the copper to the iron losses at full load, be the same for all transformers of the series as well as the ratio K/I of copper to iron loss per cm^3 at full load, both however, diminishing slightly in the same proportion as the capacity increases ; or

$$z = \text{Const.}\ K = K_0(1-mP_2)\ I = I_0(1-mP_2). \qquad \text{(III.)}$$

where m is a small fraction.

Another way in which Iz/K would be constant, and one more in accordance with the practice of some manufacturers, would be for K and Iz each to be constant, I diminishing as the capacity increased, and z increasing in the same ratio ; or

$$K = \text{Const.},\ I = \frac{I_0}{1 + nP_2},\quad z = 1 + nP_2. \quad \text{(IV.)}$$

where n is a small fraction.

Again we have the full load output

$$P_2 = \tfrac{1}{2} w n_2 C_2 F \text{ (q.p.)}$$

for a non-inductive load on which the transformer would be rated, but

$$n_2 C_2 = 2Qb^2 c = 2Qb^2 \sqrt{\frac{2K}{\rho}}.$$

and

$$F = 4p\beta^2\gamma = 4p\beta^2 \sqrt{\frac{8\pi\mu I}{w \operatorname{Sin}\delta}}.$$

hence

$$P_2 = D b^2 \beta^2 \sqrt{\frac{\mu w}{\rho \operatorname{Sin}\delta} KI}$$

where D is a constant.

Substituting in equation I. for b and β, their values determined from II. and V., we get

$$\tau = M \sqrt[4]{\frac{\mu^3 w^3 \operatorname{Sin}\delta}{\rho^3} \frac{P_2^2}{KI}}$$

where M is a constant.

Now I find for the same sample of iron that

$$\frac{\mu^3 w \operatorname{Sin}\delta}{I}$$

is very nearly constant when w is constant over the range of flux densities, or of Is, commonly used in transformers, and that it increases slightly as w diminishes.

Taking it as constant, we get

$$\tau = N \sqrt[4]{K} \sqrt{wP_2}$$

Hence if τ for a transformer of a given type be known, the equation

$$\frac{\tau^2}{wP_2} = \text{Const.},$$

will enable us to obtain fairly approximate values of τ for other transformers of the same type that differ in capacity and periodicity.

It is worth noting that equation V. above shows that, for equal heating or equal iron and copper losses per unit volume, the output of a transformer is proportional to $\sqrt{\frac{\mu w}{\text{Sin}\delta}}$.

This is not proportional to the square root of w or of the frequency as, when w increases, μ for the same flux density will diminish and $\text{Sin}\delta$ will increase.

Most Efficient Shapes of Transformers.

56. It has been shown (§ 21) that when consideration of leakage is neglected, the measure of excellence of a transformer is

$$\frac{\tau}{\text{Sin}\delta};$$

hence the most efficient transformer of a given type and capacity and made of similar iron will be that one for which τ is a maximum.

If a, α be the total cross sections (insulation, etc., included) of the copper and iron circuits respectively, and l, λ their mean lengths, then

$$g\frac{a\alpha}{l\lambda}=\tau,$$

$$\frac{al}{\alpha\lambda}=z\frac{p\text{I}}{\text{Q}\kappa}=z',$$

where g and z' are constants.

Hence, as

$$\tau z'=g\frac{a^2}{\lambda^2},\quad \frac{\tau}{z'}=g\frac{\alpha^2}{l^2}$$

for τ to be a maximum,

$$\frac{\lambda}{a} \text{ and } \frac{l}{\alpha}$$

must both be minima, and as the output

$$\text{P}_2=ha\alpha,$$

where h is a constant, as the flux and current densities will be fixed, the problem resolves itself into finding values for the dimensions of the carcass that will make

$\frac{\lambda}{a}$ and $\frac{l}{\alpha}$ both minima when $a\alpha$ is constant.

Specifying the dimensions of a shell transformer in the usual way (window$=2b$, $2b'$, tongue$=2\beta$, $2\beta'$),

$$a=4bb',\quad l=4(\beta+\beta'+2b),$$

$$\alpha=4\beta\beta',\quad \lambda=4(b+b'+\beta).$$

and proceeding by the method of indeterminate multipliers (A, B, C),

$$\mathrm{A}d\left(\frac{\lambda}{a}\right)+\mathrm{B}d\left(\frac{l}{a}\right)+\mathrm{C}d(a\alpha)=0$$

in which the coefficients of db, db', $d\beta$, and $d\beta'$ being equated to zero give us,

$$\frac{b'+\beta}{b^2b'}\mathrm{A}-\frac{2}{\beta\beta'}\mathrm{B}+b'\beta\beta'\mathrm{C}=0$$

$$\frac{b+\beta}{bb'^2}\mathrm{A}\qquad\qquad+b\beta\beta'\mathrm{C}=0$$

$$-\frac{1}{bb'}\mathrm{A}+\frac{\beta'+2b}{\beta^2\beta'}\mathrm{B}+bb'\beta'\mathrm{C}=0$$

$$\frac{\beta+2b}{\beta\beta'^2}\mathrm{B}+bb'\beta\mathrm{C}=0.$$

Eliminating A, B, and C from any two sets of three of these equations we get the two relations

$$\left.\begin{aligned}\beta(3b-b')&=2b(b'-2b)\\ \beta(\beta'-2\beta)&=b(3\beta-\beta')\end{aligned}\right\}\qquad\text{(I.)}$$

which show that $b'>2b$ and $<3b$

and $\beta'>2\beta$ and $<3\beta$.

Let $b'=\xi b$, $\beta'=\eta\beta$, $\beta=ub$,

and equations I. can be put in the forms

$$\left.\begin{aligned}u&=2\frac{\xi-2}{3-\xi}=\frac{3-\eta}{\eta-2}\\ \text{or }\xi&=\frac{3u+4}{u+2},\quad\eta=\frac{2u+3}{u+1}\end{aligned}\right\}\qquad\text{(II.)}$$

by means of which the equation of the losses

$$\frac{al}{a\lambda}=z\frac{p\mathrm{I}}{\mathrm{QK}}$$

becomes

$$\frac{(3u+4)(3u^2+6u+2)}{u^2(2u+3)(u^2+6u+6)}=z\frac{p\mathrm{I}}{\mathrm{QK}},\qquad\text{(III.)}$$

from which u (the one positive root) can be determined by trial when $zp\mathrm{I}/\mathrm{QK}$ is known. ξ and η are found from u by equations II., and so the shapes of window and tongue and their relative sizes are determined.

The relation $\frac{1}{2}wn_2\mathrm{C}_2\mathrm{F}=\mathrm{P}_2$

can now be reduced to

$$b^4u^2\xi\eta=\frac{\mathrm{P}_2}{4p\mathrm{Q}c_2\gamma w}\qquad\text{(IV.)}$$

from which b, and hence the transformer, is determined.

The equation for τ can be put in the form

$$\tau = \frac{\pi\mu P_2}{2\rho c_2\gamma} \frac{1}{(\beta+\beta^1+2b)(b+b^1+\beta)}$$

$$\text{or } \tau = \frac{\pi\mu P_2}{2\rho c_2\gamma b^2} \frac{1}{(1+\xi+u)(2+u+\eta u)} \qquad \text{(V.)}$$

by means of which it can be quickly calculated, and it will be found that the result is a true maximum.

For example, assuming the same data for design as are adopted in §§ 41 and 52,

$$z\frac{pI}{QK} = 1.029,$$

and equation III. gives

$$u = 1.1,$$

hence by means of II. we find that $b' = 2.35b$, $\beta' = 2.48\beta$, which with $\beta = 1.1b$, give the most efficient shape for a shell transformer in which $zpI/QK = 1.029$.

If $P_2 = 12.5$ K.W., the same capacity as that of the transformers in § 41, equation IV gives

$$b = 4.55,$$

and equation V.,

$$\tau = 7300.$$

The losses being al QK and $a\lambda p$I, we find that each is equal to 181 watts, so that the efficiency at full load is 97.2 per cent.

This maximum efficiency transformer will not have such good regulation on inductive loads as others less efficient, but with relatively wider windows. A compromise between efficiency and regulation can always be made suitable to the nature of the work the transformer is intended for.

For the above transformer, if wound in five sections, $x_1 + x_2 = .00075$; and the regulation would be, for a non-inductive load, 1.55 per cent., and for an inductive load of .8 power factor, 3.7 per cent. These figures can be compared with those in § 52.

57. A core transformer of the H type, in which the magnetic circuit is rectangular (2β, $2\beta'$) in section and the coils rectangular in plan, is exactly the same in geometrical shape as a shell transformer, but the copper and iron circuits of the former occupy the places of the iron and copper circuits of the latter.

Let $2b$, $2b'$ be the dimensions of the rectangular windows, or winding apertures in the laminae, the coils being wound round the $2b'$ dimension, 2β the width of the iron strip, and $2\beta'$ the dimension of the core measured perpendicular to the laminae, then

$$a=4bb', \quad l=4(\beta+\beta'+b),$$
$$\alpha=4\beta\beta', \quad \lambda=4(b+b'+2\beta).$$

and we find as in § 56, or by simply interchanging β and b, β' and b' in I., § 56, that for maximum τ, that is maximum efficiency

$$b(3\beta-\beta')=2\beta(\beta'-2\beta),$$
$$b(b'-2b)=\beta(3b-b').$$

If $b'=\xi b$, $\beta'=\eta\beta$, and $\beta=ub$ as before,

$$u=\frac{\xi-2}{3-\xi}=\frac{1}{2}\frac{3-\eta}{\eta-2},$$

$$\xi=\frac{3u+2}{u+1}, \quad \eta=\frac{4u+3}{2u+1}; \qquad \text{(II.)}$$

and the equation of the losses is

$$\frac{(3u+2)(6u^2+6u+1)}{u^2(4u+3)(2u^2+6u+3)}=z\frac{p\mathrm{I}}{\mathrm{QK}}$$

provided the coils are wound in a number of alternate layers so that the mean lengths of the primary and secondary turns are equal.

From this equation u can be found, and thence by II., ξ and η.

The equation of the output (see § 56, IV.)

$$b^4u^2\xi\eta=\frac{\mathrm{P}_2}{4p\mathrm{Q}c_2\gamma w}$$

gives b, which with u, ξ and η, determine the transformer.

In this case

$$\tau=\frac{\pi\mu\mathrm{P}_2}{2\rho c_2\gamma b^2}\,\frac{1}{(1+\xi+2u)\,(1+u+u\eta)}$$

For example, if

$$z\frac{p\mathrm{I}}{\mathrm{QK}}=1.029,$$

$\mathrm{P}_2=12.5$ K.W. as before,

then $u=.876$, $\xi=2.47$, $\eta=2.36$, $b=5.1$,

and $\tau=7320$, just the least thing better than the maximum efficiency transformer of the shell type.

If $zpI/QK=1$, max. τ would be the same for both types, and if $zpI/QK<1$, the shell type would be the better.

Magnetic leakage is in general less, and good regulation more easy to attain in core transformers than in shell transformers. To enable a comparison to be made with the shell transformer in the last paragraph, we will determine the sum of the leakage coefficients and the regulation for different kinds of load of the core transformer considered above, supposing it to be wound (*a*) in three layers, one primary and two secondary or *vice versa*; (*b*) in five layers, two primary and three secondary or *vice versa.*

From § 54,

$$x_1+x_2=\frac{1}{ij}\,\frac{\lambda^2}{3\mu\lambda'^2}\,\frac{zI}{QK},$$

and $\lambda=4(b+b'+2\beta)$, $\lambda'=4b'$, so that, using the same values for the constants as before, we find,

for (*a*) $x_1+x_2=.000381$,

(*b*) $x_1+x_2=.000127$,

from which, proceeding as in § 52, we find for the regulation

Power Factor.	Drop per cent.	
	(*a*) Three layers.	(*b*) Five layers.
1.0	1.49	1.47
.8	2.43	1.58
.6	2.55	1.43

58. It is obvious that in core transformers of the ring type in which the winding is continuous all round, the maximum efficiency shape will, other things being equal, be that in which the magnetic circuit is shortest, that is when the opening in the laminae is filled with the copper circuits. The ring type is not suitable for practical construction, but a near approach to it is the Burnand transformer,* in which the magnetic circuit is formed of square laminae from which a symmetrically placed inner square has been removed to give the winding space. Each side of the square is built and wound separately with triangular

* See "Electrician," Sept. 19, 1902.

shaped windings, and the four sides jointed together to form the completed transformer.

Let us determine the proportions of such a transformer so that τ, and hence the efficiency, shall be a maximum.

Let $2b$, $2b$, be the square opening in the laminae, 2β, $2\beta'$ the cross section of the magnetic circuit, 2β being measured in the planes of the laminae, then,

$$a=4b^2, \quad l=4(\beta+\beta'+\tfrac{2}{3}b)$$

$$\alpha=4\beta\beta', \quad \lambda=8(b+\beta)$$

Proceeding as in § 56 we find, in order that

$$\frac{\lambda}{a} \text{ and } \frac{l}{\alpha} \text{ shall be minima}$$

when $a\alpha$ is constant,

that $\quad 2\beta(\beta'-2\beta)=b(3\beta-\beta')$

and if $\quad \beta'=\eta\beta, \quad \beta=ub$ as before,

$$u=\frac{1}{2}\frac{3-\eta}{\eta-2}, \quad \eta=\frac{4u+3}{2u+1},$$

and the equation of the losses

$$\frac{al}{\alpha\lambda}=z\frac{pI}{QK}$$

becomes

$$\frac{18u^2+16u+2}{6u^2(4u^2+7u+3)}=z\frac{pI}{QK},$$

from which, for any given values of z, p, Q, I and K, u can be found and hence η.

The equation of the output,

$$b^4u^2\eta=\frac{P_2}{4pQc_2\gamma w},$$

gives b, which with u and η, determine the transformer.

For example, if we take as before

$$z\frac{pI}{QK}=1.029, \quad P_2=12.5 \text{ K.W.}$$

we find

$$u=.577, \quad \eta=2.464,$$

$$b=7.8, \quad \beta=4.5, \quad \beta'=11.09,$$

and the value of τ is 7680, which is considerably larger and hence better than for either of the two preceding types.

Iron loss = copper loss = 176.7 watts.

Efficiency = 97.26 per cent.

These transformers are wound in five or seven layers and their regulation is of a very high order. The formula in § 54 would only enable us to obtain a very rough approximation to $x_1 + x_2$ for this type.

General Solution of the Transformer Problem by a Vector Method.

Explanatory.

59. (*a*) If α be any vector representing e.m.f., current, or flux, on the plane alternate current diagram (Fig. 2) and if we understand by

$\iota\alpha$

the vector got by rotating α through a right angle in the positive direction, and hence if we understand by

$(\text{Cos}\theta + \iota\text{Sin}\theta)\alpha$ or $e^{\iota\theta}\alpha$

the vector got by rotating α through the angle θ in the positive direction, then it is well-known that operators such as $e^{\iota\theta}$ can be manipulated as ordinary algebraic symbols, and that ι can be treated as if it were the algebraic imaginary $\sqrt{-1}$.*

(*b*) If a_1, a_2, a_3 etc., be numerical multipliers, then the vector

$$\left\{ a_1 e^{\iota\theta_1} + a_2 e^{\iota\theta_2} + a_3 e^{\iota\theta_2} + \quad \right\} \alpha,$$

or the resultant or sum of the vectors

$$a_1 e^{\iota\theta_1}\alpha,\ a_2 e^{\iota\theta_2}\alpha,\ a_3 e^{\iota\theta_3}\alpha, \text{ etc.}$$

is $= \left\{ \Sigma a \text{Cos}\theta + \iota \Sigma a \text{Sin}\theta \right\} \alpha$

$$= \text{A}(\text{Cos}\psi + \iota\text{Sin}\psi)\alpha = \text{A}e^{\iota\psi}\alpha$$

where

$$\text{A}^2 = (\Sigma a \text{Cos}\theta)^2 + (\Sigma a \text{Sin}\theta)^2$$
$$= \Sigma a^2 + 2\Sigma a_1 a_2 \text{Cos}(\theta_1 - \theta_2)$$

and

$$\tan\psi = \frac{\Sigma a \text{Sin}\theta}{\Sigma a \text{Cos}\theta},$$

hence the operator

$$a_1 e^{\iota\theta_1} + a_2 e^{\iota\theta_2} + a_3 e^{\iota\theta_3} + \text{etc.} = \text{A}e^{\iota\psi}$$

where A and ψ are given by the above equations.

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* Lyle. *Alternate Current Problems.* "Electrician," 41, pp. 816-818; 42, pp. 72-74 and 148-151, 1898.

(*c*) If a represent the harmonically varying quantity $n\text{Cos}wt$, then since

$$\frac{d}{dt}(n\text{Cos}wt)=wn\text{Cos}\left(wt+\frac{\pi}{2}\right)$$

$we^{\iota\frac{\pi}{2}}a$ or $w\iota a$ will represent $\frac{d}{dt}(n\text{Cos}wt)$, and we may write

$$\frac{d}{dt}a=we^{\iota\frac{\pi}{2}}a=w\iota a.$$

60. If $\sigma/4\pi$ be the permeance of the magnetic circuit, closed or open, and limited in section by the iron core where the latter exists; and if δ be the angle of magnetic lag of the iron, then as the flux density remains very nearly constant throughout the range of operation of a transformer, we may without much error consider σ and δ as constants.

The total number N_1 of magnetic lines looped on the n_1 turns of the primary coil is the sum of three sets, namely,

1. Those traversing the iron core, produced by the magnetising ampere turns $\overline{n_1C_1+n_2C_2}$, and behind them in phase by the angle δ.

Hence these

$$=\sigma e^{-\iota\delta}(\overline{n_1C_1+n_2C_2})$$

2. Those produced by $\bar{C}_1$ and in phase with it that miss the iron core.

Let these

$$=x_{11}\sigma n_1\bar{C}_1.$$

3. Those produced by $\bar{C}_2$ and in phase with it that miss the iron core.

Let these

$$=x_{21}\sigma n_2\bar{C}_2.$$

Hence

$$N_1=n_1\sigma(e^{-\iota\delta}+x_{11})\bar{C}_1+n_2\sigma(e^{-\iota\delta}+x_{21})\bar{C}_2$$

similarly

$$N_2=n_1\sigma(e^{\iota\delta}+x_{12})\bar{C}_1+n_2\sigma(e^{-\iota\delta}+x_{22})\bar{C}_2$$

where x_{22} and x_{12} have similar significations with regard to the secondary coil that x_{11} and x_{21} have with regard to the primary.

We thus have four leakage coefficients and it will be noticed that they are connected with the two coefficients x_1 and x_2 hitherto used by the equations

$$x_1 = x_{11} - x_{21}$$
$$x_2 = x_{22} - x_{12} \text{ (see § 43).}$$

61. The equations of motion are

$$\bar{E}_1 = r_1\bar{C}_1 + n_1\frac{d}{dt}\bar{N}_1 = r_1\bar{C}_1 + wn_1e^{\iota\frac{\pi}{2}}\bar{N}_1 \qquad \text{(I.)}$$

$$\bar{E}_2 = -r_2\bar{C}_2 - n_2\frac{d}{dt}\bar{N}_2 = -r_2\bar{C}_2 - wn_2e^{\iota\frac{\pi}{2}}\bar{N}_2 \qquad \text{(II.)}$$

where $\bar{E}_1$, $\bar{E}_2$, are the terminal e.m.f's, and r_1, r_2, the internal resistances of the coils.

If R be the external resistance or its equivalent in the secondary circuit, and Cosϕ the power-factor of the load,

$$\bar{E}_2\text{Cos}\phi = \text{R}e^{\iota\phi}\bar{C}_2 \qquad \text{(III.)}$$

Eliminating E_2 between equations (II.) and (III.) and putting

$$\frac{wn_1^2\sigma}{r_1} = \tau_1 \qquad \frac{wn_2^2\sigma}{r_2} = \tau_2$$

$$\frac{wn_2^2\sigma}{\text{R}}\text{Cos}\phi = \theta$$

[Note that the θ here is the same as the θCosϕ in the early part of this paper.]

we get

$$(1 + x_{12}e^{\iota\delta})n_1\bar{C}_1 = -\left(1 + x_{22}e^{\iota\delta} + \frac{1}{\tau_2}e^{-\iota(\frac{\pi}{2}-\delta)} + \frac{1}{\theta}e^{-\iota(\frac{\pi}{2}-\delta-\phi)}\right)n_2\text{C}_2 \qquad \text{(IV.)}$$

from which by § 59, *b*, we find that

$$\frac{n_1\text{C}_1}{\Delta} = \frac{n_2\text{C}_2}{\theta\text{X}_{12}} \qquad \text{(V.)}$$

where

$$\Delta^2 = \theta^2\left(1 + 2x_{22}\text{Cos}\delta + 2\frac{\text{Sin}\delta}{\tau_2} + x_{22}^2 + \frac{1}{\tau_2^2}\right)$$
$$+ 2\theta\left\{\text{Sin}(\delta+\phi) + x_{22}\text{Sin}\phi + \frac{\text{Cos}\phi}{\tau_2}\right\} + 1,$$

$$\text{X}_{12}^2 = 1 + 2x_{12}\text{Cos}\delta + x_{12}^2;$$

and that

$$\tan\beta = \frac{\text{Cos}(\delta+\phi) + x_{12}\text{Cos}\phi + \theta\left\{(x_{12}-x_{22})\text{Sin}\delta + \dfrac{\text{Cos}\delta}{\tau_2} + \dfrac{x_{12}}{\tau_2}\right\}}{\text{Sin}(\delta+\phi) + x_{12}\text{Sin}\phi + \theta\left\{1 + (x_{12}+x_{22})\text{Cos}\delta + \dfrac{\text{Sin}\delta}{\tau_2} + x_{12}x_{22}\right\}}$$

where $\pi - \beta$ is the angle that $\overline{C}_2$ is behind $\overline{C}_1$ in phase.

62. Eliminating $\overline{C}_2$ from equations (I.) and (IV.) and putting

$$x_{11} - x_{21} + x_{22} - x_{12} = X,$$

$$\frac{1}{\tau_1} + \frac{1}{\tau_2} = T,$$

$$x_{11}x_{22} - x_{12}x_{21} - \frac{1}{\tau_1\tau_2} = m,$$

$$\frac{x_{11}}{\tau_2} + \frac{x_{22}}{\tau_1} = n,$$

we get

$$\frac{n_1\overline{E}_1}{r_1\tau_1} = \frac{e^{\iota(\phi-\delta)} + x_{11}e^{\iota\phi} + \frac{1}{\tau_1}e^{-\iota(\frac{\pi}{2}-\phi)}}{e^{-\iota(\frac{\pi}{2}-\phi)} + \theta\left\{e^{-\iota\delta} + x_{22} + \frac{1}{\tau_2}e^{-\iota\frac{\pi}{2}}\right\}} n_1\overline{C}_1$$

$$+ \frac{\theta\left\{Xe^{\iota(\frac{\pi}{2}-\delta)} + Te^{-\iota\delta} + me^{\iota\frac{\pi}{2}} + n\right\}}{e^{-\iota(\frac{\pi}{2}-\phi)} + \theta\left\{e^{-\iota\delta} + x_{22} + \frac{1}{\tau_2}e^{-\iota\frac{\pi}{2}}\right\}} n_1\overline{C}_1$$

from which by § 59, *b*, we find that

$$\frac{n_1C_1}{\Delta} = \frac{n_1E_1}{r_1\tau_1} \cdot \frac{1}{D} \qquad \text{(VI.)}$$

where

$$D^2 = 1 + 2x_{11}\text{Cos}\delta + 2\frac{\text{Sin}\delta}{\tau_1} + x_{11}{}^2 + \frac{1}{\tau_1{}^2} + 2\theta\Big\{X\text{Sin}\phi + T\text{Cos}\phi$$

$$+ \left(x_{11}X + \frac{T}{\tau_1}\right)\text{Sin}(\delta+\phi) + \left(x_{11}T - \frac{X}{\tau_1}\right)\text{Cos}(\delta+\phi) + n\text{Cos}(\delta-\phi)$$

$$- m\text{Sin}(\delta-\phi) + \left(x_{11}m + \frac{n}{\tau_1}\right)\text{Sin}\phi + \left(x_{11}n - \frac{m}{\tau_1}\right)\text{Cos}\phi\Big\}$$

$$+ \theta^2\{X^2 + T^2 + m^2 + n^2\}$$

also, if α be the angle that $\overline{C}_1$ is behind $\overline{E}_1$ in phase, so that Cosα is the power factor of the transformer,

$$D\triangle\text{Cos}a=\text{Sin}\delta+\frac{1}{\tau_1}+\theta\Big\{\text{Cos}\phi+\frac{2\text{Sin}(\delta+\phi)}{\tau_1}+(x_{12}+x_{21})\text{Cos}(\delta+\phi)$$

$$+2\Big(x_{22}\text{Sin}\phi+\frac{\text{Cos}\phi}{\tau_2}\Big)\text{Sin}\delta+2\frac{x_{22}}{\tau_1}\text{Sin}\phi+\Big(x_{12}x_{21}+\frac{2}{\tau_1\tau_2}\Big)\text{Cos}\phi\Big\}$$

$$+\theta^2\Big\{T+\Big[(x_{22}-x_{12})\ (x_{22}-x_{21})+\frac{2}{\tau_1\tau_2}+\frac{1}{\tau_2^{\,2}}\Big]\text{Sin}\delta+$$

$$\Big[\frac{2x_{22}}{\tau_1}+\frac{x_{21}+x_{12}}{\tau_2}\Big]\text{Cos}\delta+\frac{x_{22}^{\,2}}{\tau_1}+\frac{1}{\tau_1\tau_2^{\,2}}+\frac{x_{12}x_{21}}{\tau_2}\Big\}=$$

$$Q=q_0+q_1\theta+q_2\theta^2(\text{say}). \qquad \text{(VII.)}$$

The power P_1 taken in by the transformer on the primary side being

$$=\tfrac{1}{2}E_1C_1\text{Cos}a$$

we find

$$P=\tfrac{1}{2}\frac{E^2_{\ 1}}{r_1\tau_1}\ \ \frac{Q}{D^2}. \qquad \text{(VIII.)}$$

63. From equations (V.) and (VI.) we get

$$\frac{n_2C_2}{\theta X_{12}}=\frac{n_1E_1}{r_1\tau_1}\ \frac{1}{D} \qquad \text{(IX.)}$$

and as $E_2\text{Cos}\phi=RC_2$ and $\theta=\frac{wn_2^{\,2}\sigma}{R}\text{Cos}\phi$,

we find that

$$E_2=\frac{n_2}{n_1}\ \frac{X_{12}}{D}E_1. \qquad \text{(X.)}$$

As the output $P_2=\frac{1}{2}E_2C_2\text{Cos}\phi$ we find that, substituting for E_2 and C_2, that

$$P_2=\tfrac{1}{2}\frac{E_1^{\ 2}}{r_1\tau_1}\ \frac{X_{12}^{\ \ 2}}{D^2}\theta\text{Cos}\phi. \qquad \text{(XI.)}$$

64. Equation (IV.) of § 61 can be written in the form,

$$(1+x_{12}e^{\iota\delta})\ \overline{(n_1C_1+n_2C_2)}=-\Big\{(x_{22}-x_{12})e^{\iota\delta}+\frac{1}{\tau_2}e^{-\iota\left(\frac{\pi}{2}-\delta\right)}+$$

$$\frac{1}{\theta}e^{-\iota\left(\frac{\pi}{2}-\delta-\phi\right)}\Big\}n_2\bar{C}_2$$

but $\overline{n_1C_1+n_2C_2}=\overline{F/\sigma}$,

and, by § 59, *b*, we find that

$$\frac{F/\sigma}{M}=\frac{n_2C_2}{\theta X_{12}} \qquad \text{(XII.)}$$

where

$$M^2 = 1 + 2\theta\left\{(x_{22}-x_{12})\text{Sin}\phi + \frac{1}{\tau_2}\text{Cos}\phi\right\} + \theta^2\left\{(x_{22}-x_{12})^2 + \frac{1}{\tau_2^{\,2}}\right\}$$

and $X^2_{12} = 1 + 2x_{12}\text{Cos}\delta + x_{12}^{\,2}$ (as before).

Combining equations (XII.) and (IX.), we have

$$\frac{F}{\sigma M} = \frac{n_1 E_1}{r_1\tau_1}\,\frac{1}{D} \qquad \text{(XIII.)}$$

and as the iron loss (see § 14),

$$H_3 = \tfrac{1}{2} w \frac{F^2}{\sigma}\text{Sin}\delta,$$

we find by means of equation (XIII.), that

$$H_3 = \tfrac{1}{2}\frac{E_1^{\,2}}{r_1\tau_1}\,\frac{M^2}{D^2}\text{Sin}\delta. \qquad \text{(XIV.)}$$

65. The primary copper loss H_1 being

$$= \tfrac{1}{2} r_1 C_1^{\,2}$$

we find by equation (VI.), § 62, that

$$H_1 = \tfrac{1}{2}\frac{E_1^{\,2}}{r_1\tau_1^{\,2}}\,\frac{\Delta^2}{D^2}\ ,$$

and the secondary copper loss H_2 being

$$= \tfrac{1}{2} r_2 C_2^{\,2}\,;$$

also we find by equation (IX.), § 63, that

$$H_2 = \tfrac{1}{2}\frac{E_1^{\,2}}{r\tau_1\tau_2}\,\frac{\theta^2 X_{12}^{\,2}}{D^2}\ .$$

66. The efficiency

$$\eta = \frac{P_2}{P_1} = \frac{X^2_{12}\theta\text{Cos}\phi}{Q}$$

$$= X^2_{12}\text{Cos}\phi\frac{\theta}{q_0 + q_1\theta + q_2\theta^2} \quad \text{(see §§ 62, 63),}$$

is a maximum when

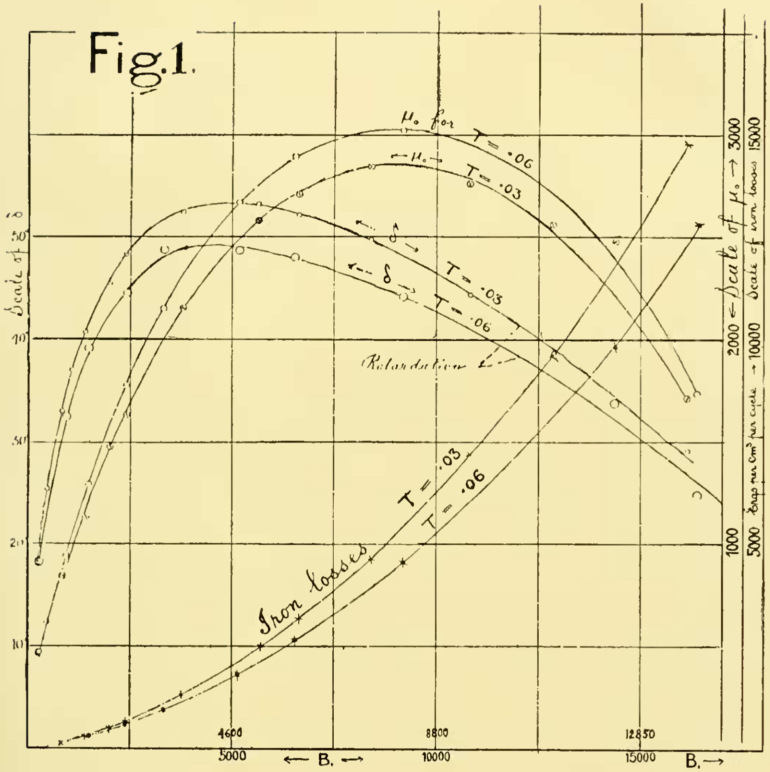
$$\theta^2 = \frac{q_0}{q_2} \quad \text{(see § 15),}$$

and its maximum value is

$$\frac{X_{12}^{\,2}\text{Cos}\phi}{q_1 + 2\sqrt{q_0 q_2}}.$$

67. Thus, without making any assumptions as regards leakage, all the important variables in the general theory of the trans-

Fig.1.



μ₀ for T = ·06
← μ₀ → T = ·03
← δ → T = ·03
← δ → T = ·06
Retardation
Iron losses T = ·03
T = ·06
Scale of δ
50°
40°
30°
20°
10°
2000 ← Scale of μ₀ →
3000
1000
15000
10000
5000
Ergs per Cm³ per cycle → Scale of iron losses
4600
8800
12850
5000
10000
15000
← B. →
B. →